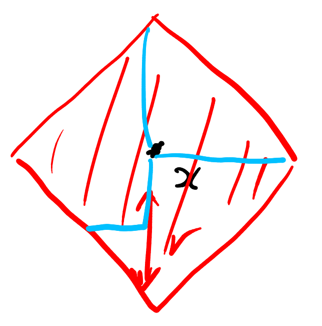
Recall: $B_r(x) = \{x' \in X : d(x, x') < r\}$.
"open ball centred at $x$ with radius $r$"

e.g. $(\mathbb{R}^2, d_1)$




$B_r(x)$ (edges are excluded)

**Def** Suppose $A$ is a set in $(X, d)$. We say $x \in A$ is an <u>interior point</u> of $A$ if there exists an open ball $B_r(x) \subseteq A$. The set of interior points is called the <u>interior</u> of $A$. We say $A$ is an <u>open set</u> if all points inside $A$ are interior points. If $A$ is an open set and $x \in A$, we say $A$ is an <u>open neighbourhood</u> of $x$.

<u>Examples:</u>

* Open balls are open sets:

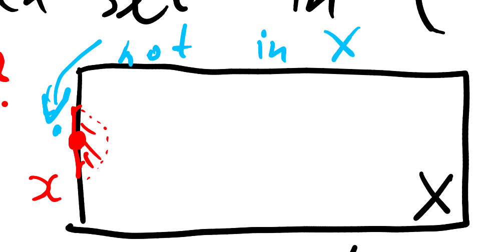

$s = r - d(x, a)$

Suppose $x' \in B_s(a)$.
Then $d(x, x') \le \underbrace{d(x,a)}_{<r} + \underbrace{d(a, x')}_{<s \, = r - d(x,a)}$

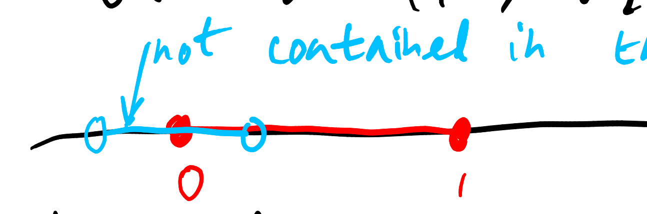
$< d(x, a) + s$
$= \cancel{d(x,a)} + r - \cancel{d(x,a)}$
$= r$

* $(0,1)$ is an open set inside $(\mathbb{R}, d_2)$.
  $\nwarrow B_{\frac{1}{2}}(\frac{1}{2})$
* $X$ is open set in $(X, d)$.

What is $B_1(x)$?



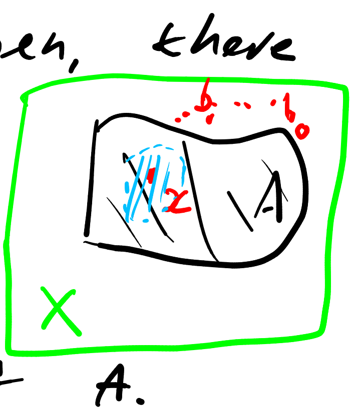

* $\emptyset$ is an open set in $(X, d)$.
* $[0,1]$ is open in $([0,1], d_2)$ but not in $(\mathbb{R}, d_2)$.




<u>Theorem</u> Let $A$ be a set in $(X, d)$. Then $A$ is open iff $A$ contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

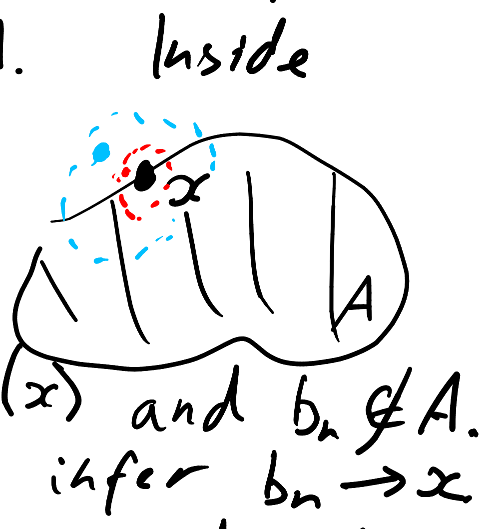
<u>Proof</u> <u>A is open $\Rightarrow$ none of boundary:</u>

Pick any $x \in A$. Since $A$ is open, there is an open ball $B_r(x) \subseteq A$. Therefore, no sequence outside of $A$ can converge to $x$. So $x$ is not a boundary point of $A$.




<u>A is not open $\Rightarrow$ some of its boundary:</u>

Since $A$ is not open, there is some point $x \in A$ s.t. every $B_r(x) \not\subseteq A$. We will show $x \in \partial A$. Inside sequence: $a_n = x$.
Outside sequence: for each radius $r_n = \frac{1}{n}$, there is a point $b_n \in B_{r_n}(x)$ and $b_n \notin A$. Since $d(b_n, x) < \frac{1}{n}$, we infer $b_n \to x$. So $x \in \partial A$. We conclude $x \in A \cap \partial A \neq \emptyset$.




$\square$

* Recall $\emptyset$ is both open and closed in $(X, d)$. Trick: $\partial \emptyset = \emptyset$
* Recall $X$ is both open and closed in $(X, d)$ $\partial X = \emptyset$
* $[0,1]$ is both open and closed inside $([0,1] \cup [2,3], d_2)$.
* $[0,1)$ is neither open nor closed inside $(\mathbb{R}, d_2)$. $\partial[0,1) = \{0, 1\}$.

<u>Theorem</u> Let $A$ be any set in $(X, d)$. Then $A$ is open iff $X \setminus A$ is closed.

<u>Proof</u> Note: $\partial A = \partial(X \setminus A)$.

$a_n \to x$ — outside
$b_n \to x$ — inside

If $A$ contains none of its boundary, then $X \setminus A$ contains all of its boundary. (Iff) $\square$