Lump sum taxes

def Consider a pure exchange economy (u, e). A feasible allocation x^* along with prices p^* form a pure exchange equilibrium with lump sum taxes t^* if

① the total taxes levied are 0, i.e. $\sum_{h \in H} t^*_h = 0$, and

② $x^*_h \in \underset{x_h \in \mathbb{R}^{\#}_{+}}{\arg\max}\ u_h(x_h)$
s.t. $p^* \cdot x_h \leq p^* \cdot e_h - t^*_h$.

Second Welfare Theorem. Consider a pure-exchange economy (u, e) in which all households' utility functions are continuous, strictly increasing and strictly quasi-concave, and aggregate endowments of each good are strictly positive. (Same conditions as the existence theorem.) If x^* is an efficient allocation, then there exists prices p^* and lump sum transfers t^* such that (x^*, p^*, t^*) form a pure-exchange equilibrium.

Proof Trick invented by Maskin and Roberts (1980).

Consider the pure exchange economy (u, x^*), i.e. after Robin Hood replaced e with x^*. By the existence theorem, there exists a pure exchange equilibrium (x^{**}, p^*).

Each household can stick to their endowment x^*_h. So
$$u_h(x^*_h) \leq u_h(x^{**}_h) \text{ for all } h \in H.$$
Since x^* is efficient, we can't have $u_h(x^*_h) < u_h(x^{**}_h)$ for any $h \in H$ – otherwise x^{**} would Pareto dominate x^*.

We conclude that
$$u_h(x^*_h) = u_h(x^{**}_h) \text{ for all } h \in H.$$
So (x^*, p^*) is an equilibrium in the (u, x^*) economy.

Let $t^*_h = p^* \cdot e_h - p^* \cdot x^*_h$, so household h can just afford to buy x^*_h after selling off their endowment e_h.

Then each household's budget constraint with transfers becomes
$$p^* \cdot x_h \leq p^* \cdot e_h - t^*_h = \cancel{p^* \cdot e_h} - [\cancel{p^* \cdot e_h} - p^* \cdot x^*_h]$$
$$\iff p^* \cdot x_h \leq p^* \cdot x^*_h$$
same as endowment in (u, x^*) economy.

Finally, we check that $\sum_h t^*_h = 0$:
$$\sum_h t^*_h = \sum_{h \in H} p^* \cdot (e_h - x^*_h)$$
$$= p^* \cdot \sum_{h \in H} (e_h - x^*_h)$$
$$= p^* \cdot 0$$
$$= 0,$$
as required.

We conclude that (x^*, p^*, t^*) is an equilibrium with lump-sum transfers. □