Infinite horizon dynamic programming

Previously: finite horizon cake-eating problem:

$$V_t(k) = \max_{c,\,k'} u(c) + \beta V_{t+1}(k') \quad \text{if } t<T$$
$$\text{s.t. } c+k'=k,$$

$V_T(k) = u(k).$

OR

$$V_t(k) = \max_{c,\,k'} \cdots \quad \text{if } t \le T$$
$$\text{s.t. } \cdots$$

and

$V_{T+1}(k) = 0.$

$$V_1(k) = \max_{c_1,\dots,c_T} \sum_{t=1}^{T} \beta^{t-1} u(c_t)$$
$$\text{s.t. } \sum_{t=1}^{T} c_t = k.$$

Infinite horizon version

$$V_1(k) = \sup_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$
$$\text{s.t. } \sum_{t=1}^{\infty} c_t = k.$$

short-hand. means $\lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} u(c_t)$

$$V_2(k) = \sup_{\{c_t\}_{t=2}^{\infty}} \sum_{t=2}^{\infty} \beta^{t-2} u(c_t)$$
$$\text{s.t. } \sum_{t=2}^{\infty} c_t = k.$$
$$= V_1(k).$$

Funny Bellman equation:

$$V_t(k) = \sup_{c,\,k'} u(c) + \beta V_{t+1}(k')$$
$$\text{s.t. } c+k'=k.$$

k': tomorrow's cake

Simplify:

$$V(k) = \sup_{c,\,k'} u(c) + \beta V(k')$$
$$\text{s.t. } c+k'=k.$$

recursive Bellman equation: same V on both sides.

Theorem: Principle of optimality

$$V(k) = \sup_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \quad \text{s.t. } \sum_{t=1}^{\infty} c_t = k$$

$$= \sup_{\{c_t\}_{t=1}^{\infty},\,k'} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \quad \text{s.t. } \sum_{t=2}^{\infty} c_t = k' \text{ and } c_1+k'=k.$$

$$= \sup_{\substack{c_1,\,k'\ge 0 \\ \text{s.t. } c_1+k'=k}} \left[\sup_{\{c_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \quad \text{s.t. } \sum_{t=2}^{\infty} c_t = k' \right]$$

$$= \sup_{\substack{c_1,\,k'\ge 0 \\ \text{s.t. } c_1+k'=k}} u(c_1) + \left[\sup_{\{c_t\}_{t=2}^{\infty}} \sum_{t=2}^{\infty} \beta^{t-1} u(c_t) \quad \text{s.t. } \sum_{t=2}^{\infty} c_t = k' \right]$$

$$= \sup_{\substack{c_1,\,k'\ge 0 \\ \text{s.t. } c_1+k'=k}} u(c_1) + \beta V(k') \qquad \square$$

Bellman operator: $F: CB(X) \to CB(X)$

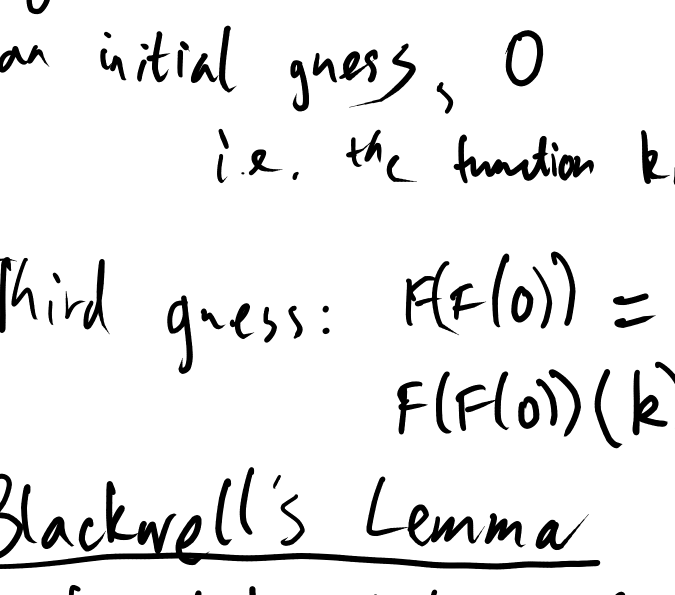
or some other set of functions

$F(V_{\text{tomorrow}}) = V_{\text{today}}.$

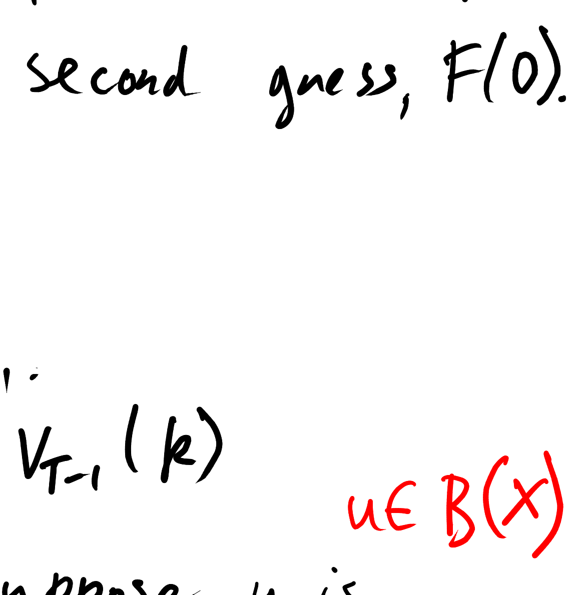
Bellman eq: $V_{\text{today}} = F(V_{\text{tomorrow}})$

e.g.: $$F(\tilde{V})(k) = \sup_{c,\,k'\ge 0} u(c) + \beta \tilde{V}(k') \quad \text{s.t. } c+k'=k,$$

$F(V) = V$, i.e. V is a fixed point of the Bellman operator F.



finite horizon: V_{T+1}
or
an initial guess, 0
i.e. the function $k \mapsto 0$



finite horizon: V_T
second guess, $F(0)$.

Third guess: $F(F(0)) = V_{T-1}$.
$F(F(0))(k) = V_{T-1}(k)$

Blackwell's Lemma Suppose u is a bounded utility function. Then the Bellman operator is a contraction of degree β on $(B(\mathbb{R}_+), d_\infty)$.

$u \in B(X)$

Proof Pick any $V \in B(\mathbb{R}_+)$. First we show $F(V)$ exists and is bounded, i.e. $F(V) \in B(\mathbb{R}_+)$. Since u and V are bounded, there exist open balls $N_r(0)$ and $N_s(0)$ that contain their ranges. Therefore, any combination of (c,k') gives a value inside $N_{r+\beta s}(0)$. So the supremum is finite, so $F(V)$ exists and is bounded.

Second, we prove F is a contraction: Consider any $V, W \in B(\mathbb{R}_+)$. Then

$$F(V)(k) = \sup_{c} u(c) + \beta V(k-c)$$
$$= \sup_{c\in[0,k]} u(c) + \underbrace{\beta W(k-c) - \beta W(k-c)}_{0} + \beta V(k-c)$$
$$\le \sup_{c\in[0,k],\ \tilde{c}\in[0,k]} u(c) + \beta W(k-c) - \beta W(k-\tilde{c}) + \beta V(k-\tilde{c})$$

if u is increasing, choose $\tilde{c}=0$

$$= \left[\sup_{c\in[0,k]} u(c) + \beta W(k-c)\right] + \left[\sup_{\tilde{c}\in[0,k]} -\beta W(k-\tilde{c}) + \beta V(k-\tilde{c})\right]$$
$$= F(W)(k) + \beta \left[\sup_{\tilde{c}\in[0,k]} V(k-\tilde{c}) - W(k-\tilde{c})\right]$$
$$\le F(W)(k) + \beta \left[\sup_{k'\ge 0} V(k') - W(k')\right]$$
$$\le F(W)(k) + \beta \left[\sup_{k'\ge 0} |V(k') - W(k')|\right]$$
$$= F(W)(k) + \beta\, d_\infty(V, W).$$

We just proved:
$$F(V)(k) \le F(W)(k) + \beta\, d_\infty(V,W).$$

Rearranging gives
$$F(V)(k) - F(W)(k) \le \beta\, d_\infty(V,W).$$

Swapping V & W in the logic above gives
$$F(W)(k) - F(V)(k) \le \beta\, d_\infty(V,W)$$
$$\Rightarrow |F(V)(k) - F(W)(k)| \le \beta\, d_\infty(V,W).$$

Since this true for all k, we deduce
$$\sup_{k\ge 0} |F(V)(k) - F(W)(k)| \le \beta\, d_\infty(V,W)$$
$$\Rightarrow d_\infty(F(V), F(W)) \le \beta\, d_\infty(V,W).$$

We conclude F is a contraction of degree β. $\square$

We know:
* F is a contraction on $(B(\mathbb{R}_+), d_\infty)$
* the value function V solves $V = F(V)$
* $(B(\mathbb{R}_+), d_\infty)$ is complete
* Banach F.P.T.: there exists a unique fixed point $V^* = F(V^*)$.
* $V^* = \lim_{n\to\infty} F^n(V_0)$. ← any guess

A legitimate form of circular reasoning:

e.g.: increasing bounded functions

* assume a space of functions (A, d_∞).
  - needs to be a subset of $B(\mathbb{R}_+)$, and
  - $F: A \to A$
* If that is satisfied, then Banach's F.P.T. says that $V^* \in A$.