$p \in \mathbb{R}_+$ price of the firm's output

$w \in \mathbb{R}_+^{N-1}$ prices of the inputs

e.g. wages

The firm's profit function: $\sum_{n=1}^{N-1} w_n x_n$

$$\pi(p; w) = \max_{x \in \mathbb{R}_+^{N-1}} \underbrace{\underbrace{pf(x)}_{\text{output}}}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{cost}}$$

$$= pf(\underbrace{x(p;w)}) - w \cdot \underbrace{x(p;w)}$$

$\boxed{\text{the}}$ factor demand function

First order condition w.r.t. x_n:

$$\underbrace{p \underbrace{\frac{\partial f(x)}{\partial x_n}}_{MP_n}}_{MRP_n} = w_n.$$

$$\frac{FOC\ x_n}{FOC\ x_m}: \quad \frac{\frac{\partial f(x)}{\partial x_n}}{\frac{\partial f(x)}{\partial x_m}} = \frac{w_n}{w_m}.$$

Detour: Chain rule (F2) and implicit function theorem (F3)

$f: \mathbb{R}^p \to \mathbb{R}^q$,

$g: \mathbb{R}^q \to \mathbb{R}^r$.

$h(x) = g(f(x))$, $h: \mathbb{R}^p \to \mathbb{R}^r$.

(*) Theorem (chain rule) If f and g are differentiable, then h is differentiable and $h'(x) = g'(f(x))\, f'(x)$.

$f: \mathbb{R} \to \mathbb{R}^2$ and $g: \mathbb{R}^2 \to \mathbb{R}$

$f(t) = (-t, \sqrt{t})$ $\quad g(x, y) = xy^2$.

$h(t) = g(f(t))$.

What is $h'(t)$?

without chain rule:

$h(t) = g(-t, \sqrt{t}) = (-t)(\sqrt{t})^2 = -t^2$.

$h'(t) = -2t$.

with chain rule:

$h'(t) = g'(f(t))\, f'(t)$

$g'(x, y) = [g_1(x,y) \quad g_2(x,y)]$

$= [y^2 \quad 2xy]$

$f'(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2\sqrt{t}} \end{bmatrix}$

$h'(t) = g'(-t, \sqrt{t})\, f'(t)$

$= [t \quad 2(-t)\sqrt{t}] \begin{bmatrix} -1 \\ \frac{1}{2\sqrt{t}} \end{bmatrix}$

$= -t + \cancel{2}(-t)\frac{\cancel{\sqrt{t}}}{\cancel{2}\cancel{\sqrt{t}}}$

$= -2t$.

Implicit Function Theorem

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $g: \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies

$f(x, g(x)) = 0$ for all x.

Consider (x^*, y^*). If $y = g(x^*)$ and

$$\frac{\partial f(x^*, y^*)}{\partial y} \neq 0$$

then g is differentiable with

$$g'(x^*) = -\frac{\frac{\partial f(x^*, y^*)}{\partial x}}{\frac{\partial f(x^*, y^*)}{\partial y}}.$$

"Proof": $f(x, g(x)) = 0$ for all x

$$\left[\frac{\partial f(x, g(x))}{\partial x} + \frac{\partial f(x, g(x))}{\partial y} g'(x)\right]_{(x,y) = (x^*, y^*)} = 0$$

Back to firms problem

Ex 2.1

r royalties

l_m musician labour

l_t technician labour

w_m, w_t wages

$f(l_m, l_t)$ # songs output

The music company's profit maximasition problem is:

$$\max_{l_m, l_t} r f(l_m, l_t) - w_m l_m - w_t l_t$$

Ex 2.2

w waste input

$g(w)$ glycerine output

$d(w)$ diesel output

p^w, p^g, p^d prices

The bio-energy profit function is:

$$\pi(p^g, p^d; p^w) = \max_w p^g g(w) + p^d d(w) - p^w w$$

Ex 2.3

x crude oil input

$e = f(x)$ ethylene intermediate

$y = g(e)$ plastic output

p_x, p_y prices

Integrated firm's profit function is

$$\pi(p_y, p_x) = \max_x p^y g(f(x)) - p^x x.$$