Expenditure function

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}^N_+} p \cdot x = p \cdot h(p, \bar{u})$$

$$\text{s.t. } u(x) \geq \bar{u}$$

utility target

expenditure function

Hicksian demand

Bellman equation

$$v(p, m) = \max_{\bar{u}} \bar{u}$$

$$\text{s.t. } e(p, \bar{u}) \leq m.$$

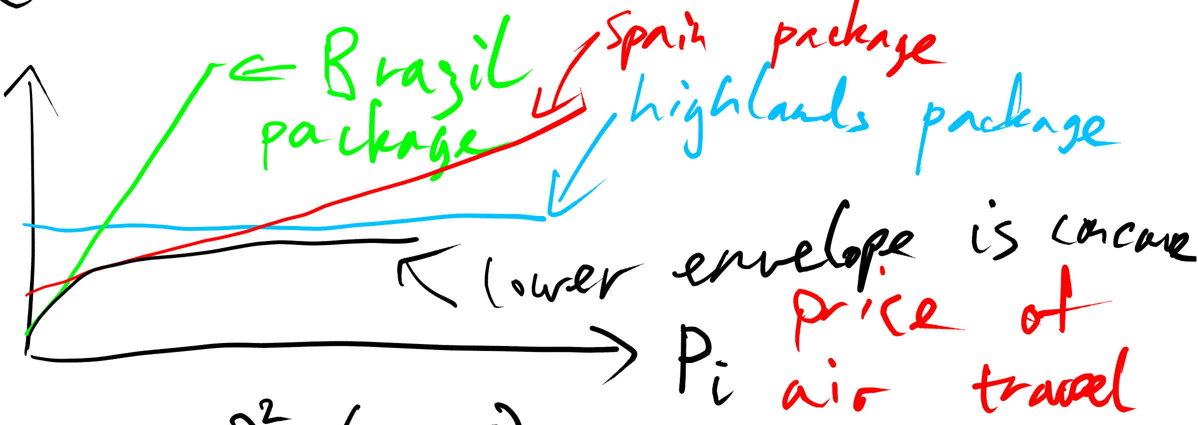
Envelope theorem:

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = h_i(p, \bar{u})$$

$$\frac{\partial e(p, \bar{u})}{\partial \bar{u}} = \mu(p, \bar{u})$$

Lagrange multiplier

Since the objective is linear in prices (and prices don't appear in the constraint), $e(p, \bar{u})$ is concave in prices.



Therefore, $\dfrac{\partial^2 e(p, \bar{u})}{\partial p_i^2} < 0$

Combining, we deduce

$$\frac{\partial h_i(p, \bar{u})}{\partial p_i} = \frac{\partial^2 e(p, \bar{u})}{\partial p_i^2} < 0.$$