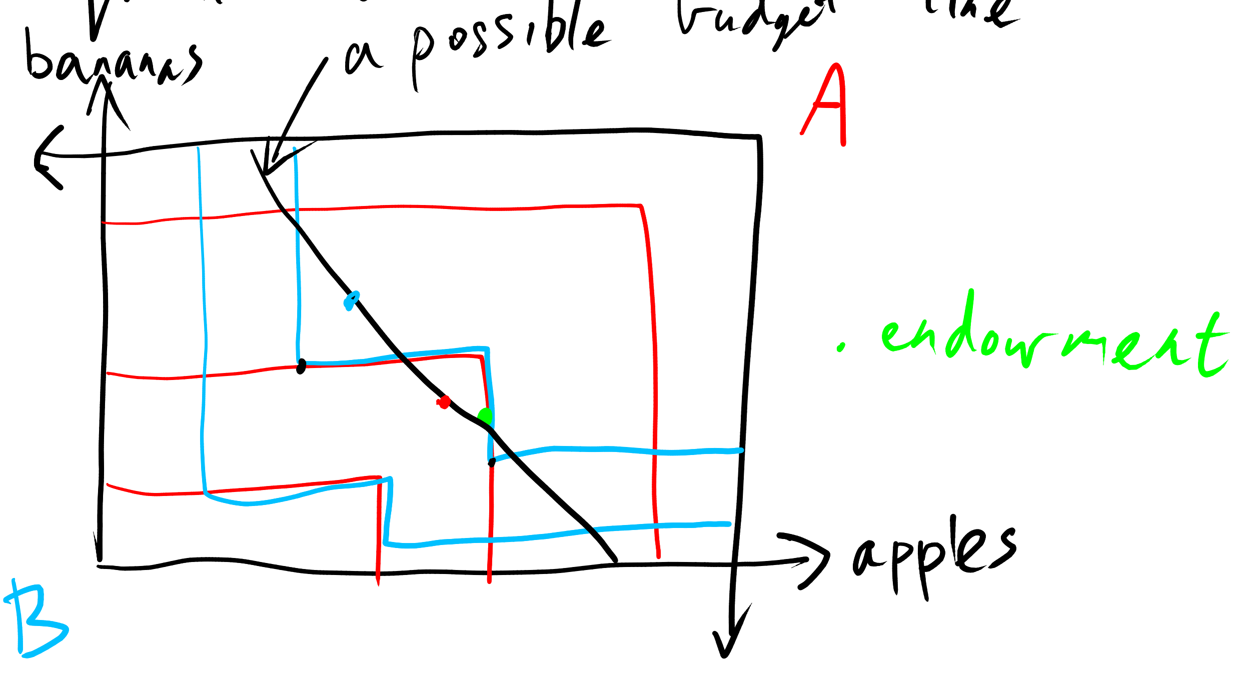
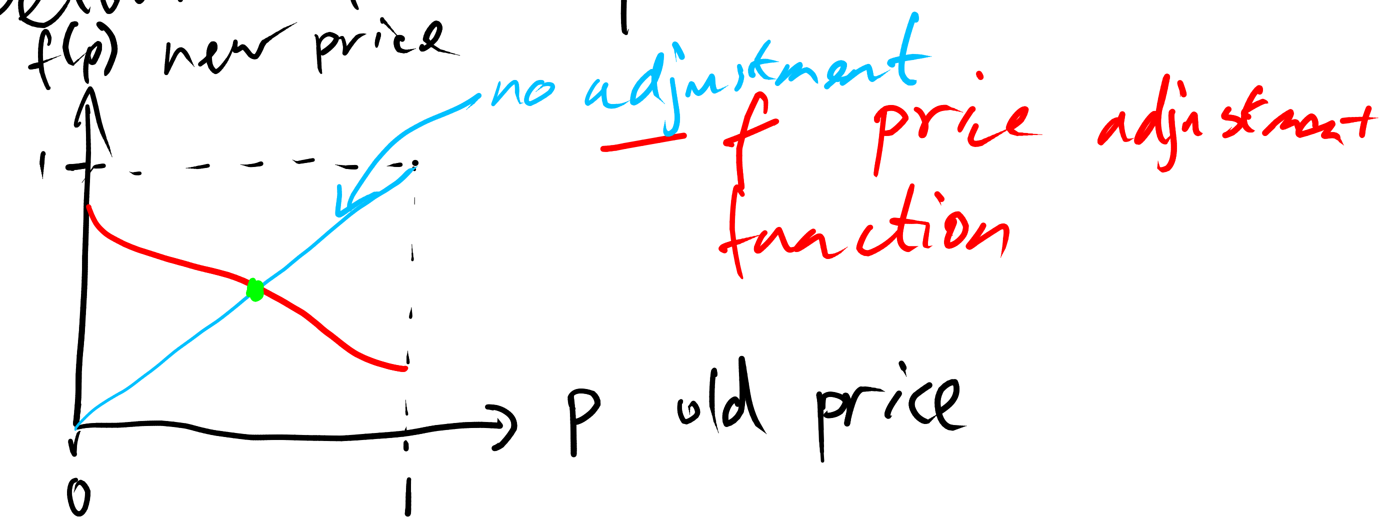
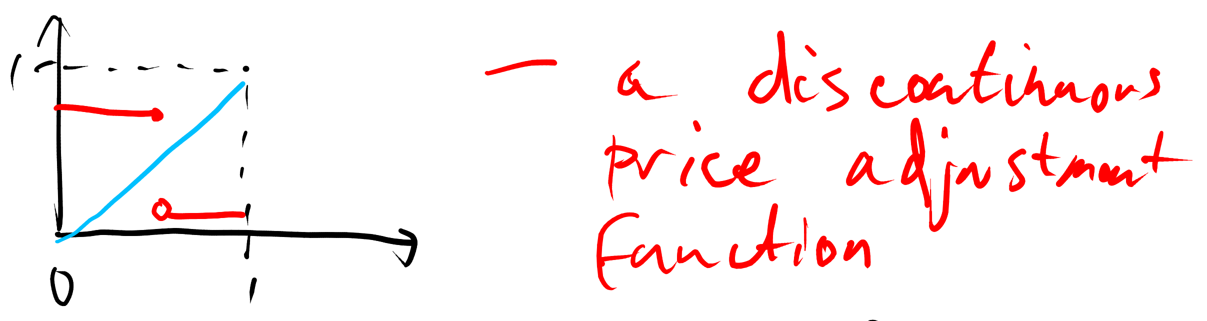
How many equilibria?

no equilibria:



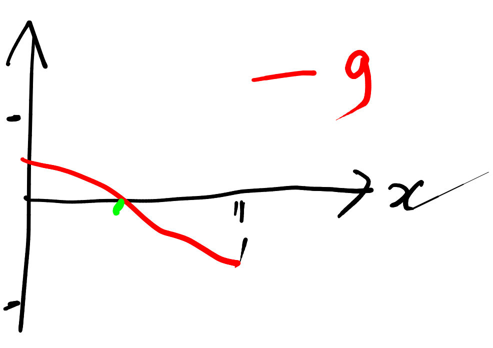
Detour: fixed points





Theorem If $f:[0,1]\to[0,1]$ is continuous, then f has a fixed point.

Proof Let $g(x)=f(x)-x$.



Notice g is continuous, x^* is a fixed point of f $\iff g(0)=0$, and $g(0)\geq 0$ and $g(1)\leq 0$.

By the intermediate value theorem there exists some x^* s.t. $g(x^*)=0$. So $f(x^*)=x^*$. ☐

Brouwer's fixed point theorem (+)

If $f:X\to X$ is a continuous function and $X\subset\mathbb{R}^N$, and X is non-empty, convex, and compact, then f has a fixed point.

Theorem Consider a pure-exchange economy (u,e) in which each utility function $u_h:\mathbb{R}^N_+\to\mathbb{R}$ is continuous, strictly increasing and strictly quasi-concave, and aggregate endowments in each good n is strictly positive, i.e. $\sum_{h\in H} e_{hn}>0$. Then there exists an equilibrium (x^*,p^*).

Proof Recall I defined $z(p)$ to be the excess demand function. Note that z is continuous (see the notes.)

Let $\bar{z}$ be the truncated excess demand function

$$\bar{z}_i(p)=\min\{1,z_i(p)\}.$$

Notice that p^* is an eq. price $\iff \bar{z}(p^*)=0$.

Let $f(p)$ be the price adjustment function

$$f_i(p)=p_i+\max\{0,\bar{z}_i(p)\}.$$

Problem: could have prices diverging to ∞.

Let $g(p)$ be the price adjustment function

$$g(p)=\frac{f(p)}{\sum_{i=1}^{N} f_i(p)}.$$

Thus g is a function with the following properties:

* $g(p)$ prices always sum to 1. So $g:[0,1]^N\to[0,1]^N$.
* g is continuous, since its ingredients ($\bar{z}$, max, min, $\frac{1}{x}$, etc.) are continuous
* p^* is a fixed point of g $\iff$ p^* is an eq. price.

By Brouwer's fixed point theorem, there exists a fixed point p^* of g, i.e. $p^*=g(p^*)$.

Moreover, p^* is an eq. price, so there exists an allocation $x^*=x(p^*)$ such that (x^*,p^*) is an equilibrium. ☐