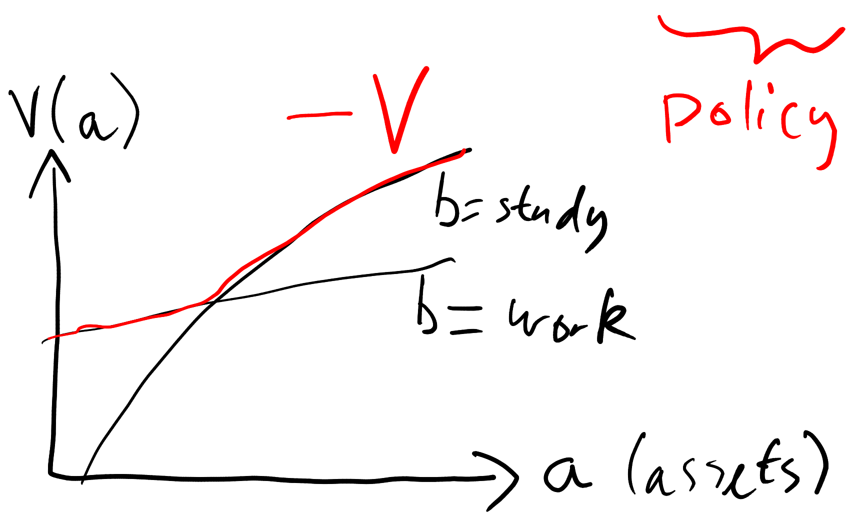
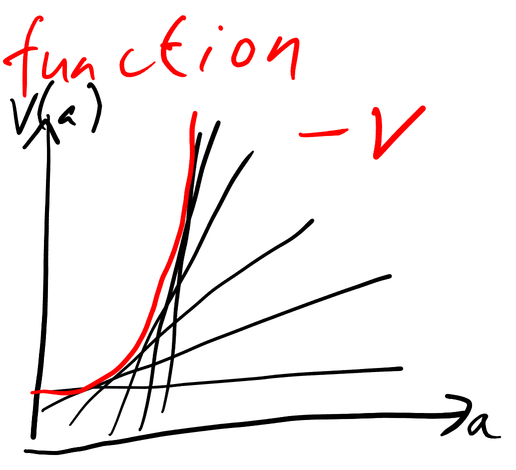
$$\pi(p;w) = \max_{x \in \mathbb{R}^{n-1}_+} pf(x) - w \cdot x$$

$$\underbrace{V(a)}_{\substack{\text{state variable} \\ \text{value function}}} = \max_{\underbrace{b}_{\text{choice variable}}} \underbrace{v(a,b)}_{\text{objective function}}$$

$$= v(a, \underbrace{b(a)}_{\text{policy function}})$$





— V also called "upper envelope"

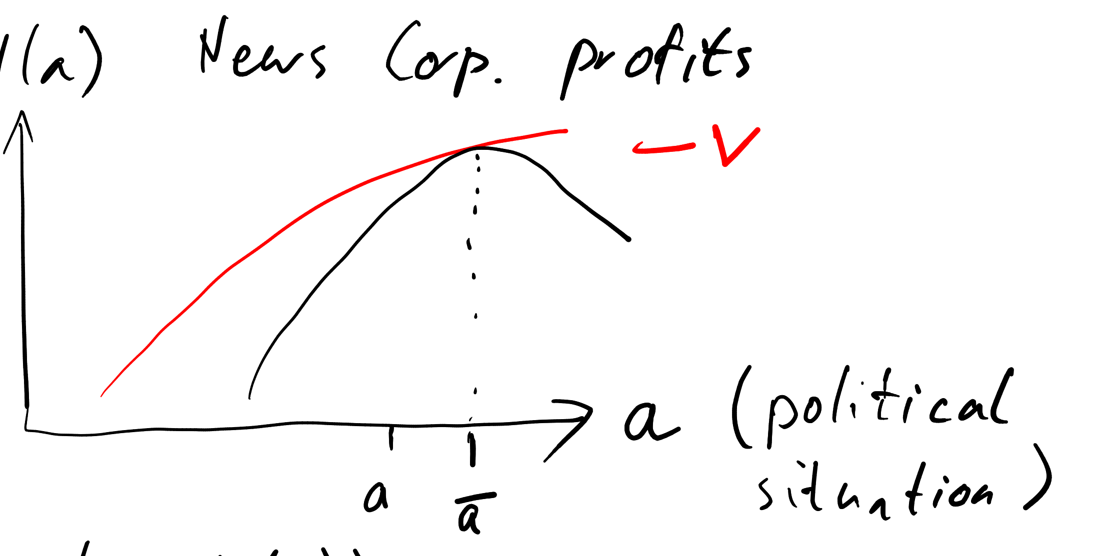
Theorem 2.1 (envelope theorem)

Let $v: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function. Let $V(a) = \max_b v(a,b)$ be its value function, $b(a)$ be an optimal policy. If V is differentiable, then

$$V'(a) = \left.\frac{\partial v(a,b)}{\partial a}\right|_{b=b(a)},$$

or $V'(a) = v_a(a, b(a))$.

Proof $V(a)$ News Corp. profits



$$L(a) = v(a, \underbrace{b(\bar{a})}_{\text{last words}})$$

$L(a) \leq V(a)$ for all a.

$L(\bar{a}) = V(\bar{a})$.

Therefore $\bar{a}$ minimises $V(a) - L(a)$.

FOC a: $V'(\bar{a}) = L'(\bar{a}) = \left.\frac{\partial v(a,b)}{\partial a}\right|_{b=b(\bar{a})}$

□

Chain rule proof:

$$V(a) = v(a, b(a)).$$

$$V'(a) = \left.\underbrace{\frac{\partial v(a,b)}{\partial a}}_{\text{direct effect}} + \overbrace{\underbrace{\frac{\partial v(a,b)}{\partial b}}_{=0, \text{ by F.O.C.}} b'(a)}^{\text{indirect effect}}\right|_{b=b(a)}$$

$$= \left.\frac{\partial v(a,b)}{\partial b}\right|_{b=b(a)}$$

□

e.g. ℓ # workers

w wage

$$\pi(w) = \max_\ell 10\sqrt{\ell} - w\ell.$$

What is $\pi'(w)$?

w/o envelope theorem

FOC ℓ: $\frac{10}{2\sqrt{\ell}} = w$

$\iff \frac{5}{w} = \sqrt{\ell}$

$\iff \ell(w) = \frac{25}{w^2}$.

$$\pi(w) = 10\sqrt{\ell(w)} - w\,\ell(w)$$

$$= 10\sqrt{\frac{25}{w^2}} - w\frac{25}{w^2} \quad \Leftarrow \text{mess!}$$

$$= \frac{50}{w} - \frac{25}{w}$$

$$= \frac{25}{w}.$$

$$\pi'(w) = -\frac{25}{w^2}$$

w/ envelope theorem

$$\pi'(w) = \left[\frac{\partial}{\partial w}\{10\sqrt{\ell} - w\ell\}\right]_{\ell=\ell(w)}$$

$$= [-\ell]_{\ell=\ell(w)}$$

$$= -\ell(w).$$

More generally:

$$\frac{\partial \pi(p;w)}{\partial p} = \left[\frac{\partial}{\partial p}\{pf(x) - w \cdot x\}\right]_{x = x(p;w)}$$

factor demand policy

$$= [f(x)]_{x=x(p;w)}$$

$$= f(x(p;w)) = \underbrace{y(p;w)}_{\text{output policy}}$$

$$\frac{\partial \pi(p;w)}{\partial w_n} = \left[\frac{\partial}{\partial w_n}\{pf(x) - w \cdot x\}\right]_{x=x(p;w)}$$

$$= [-x_n]_{x=x(p;w)}$$

$$= -x_n(p;w).$$