**Def** A <u>sequence</u> inside a set $X$ is any function with domain $\mathbb{N}$ and co-domain $X$. Sequences are often denoted as

$$x_0, \quad x_1, \quad x_2, \quad \dots$$

or $\{x_n\}_{n=0}^{\infty}$

or $x_n$.

eg: If $X = \{\text{attack}, \text{retreat}\}$, then

$$x_n = \begin{cases} \text{attack} & \text{if } n \text{ is divisible (Sunday)} \\ \text{retreat} & \text{otherwise} \end{cases}$$

is an example of a sequence.

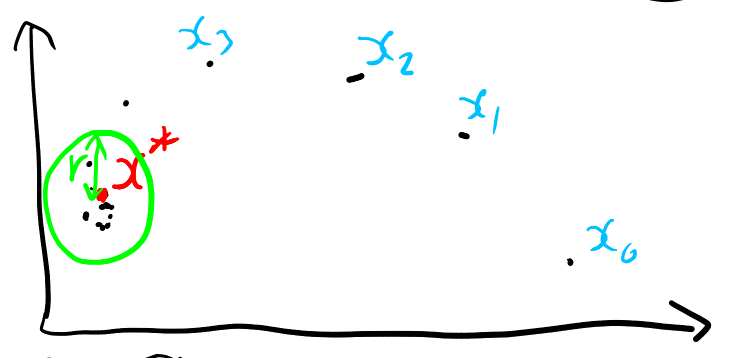
$y_n = 2n$ is a sequence inside of natural numbers $\mathbb{N}$.

**Def** Suppose $x_n$ is a sequence in a metric space $(X, d)$, i.e. $x_n \in X$. We say that $x_n$ <u>converges</u> to $x^* \in X$ $(x_n \to x^*)$ if for every radius $r > 0$, there exists a $N$ such that

$$d(x_n, x^*) < r \quad \text{for all } n \geq N.$$

$x_n \in B_r(x^*)$

$x^*$ is called the <u>limit</u> of $x_n$.

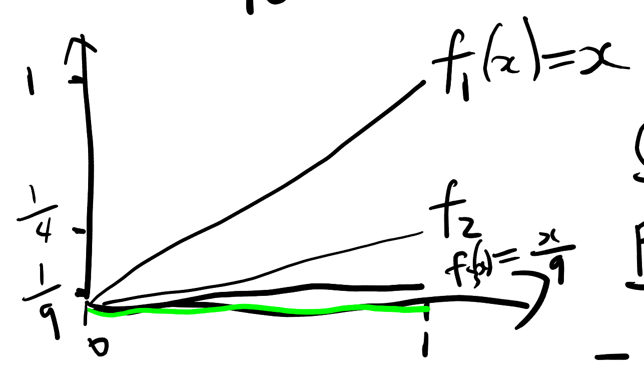



**Def** The <u>open ball</u> centred at $x$ with radius $r > 0$ in $(X, d)$ is

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

eg: $x_k = \frac{1}{k}$. Abuse of notation: $x_0 = ???$
Then $x_k \to 0$, inside $(\mathbb{R}, d_2)$

* $x_k = \frac{1}{k}$ inside $(\mathbb{R}, d)$ where $d$ is the discrete metric. $x_k$ does not converge!
Note that $d(0, x_k) = d(0, \frac{1}{k}) = 1$.
So there is no suitable $N$ for $r = \frac{1}{2}$.
* $x_k = \frac{1}{k}$ inside $(\mathbb{R}_{++}, d_2)$ does not converge. Note: $0 \notin \mathbb{R}_{++}$.
* $f_k(x) = \frac{x}{k^2}$ inside $(B[0,1], d_\infty)$




<u>Claim</u>: $f_k \to f^*$.

<u>Proof</u>: $d_\infty(f^*, f_k) = \sup_{x \in [0,1]} |f^*(x) - f_k(x)|$

$$= \sup_{x \in [0,1]} \left|0 - \frac{x}{k}\right|$$

$$= \sup_{x \in [0,1]} \frac{x}{k} = \frac{1}{k}.$$

Therefore, if we set $N = \frac{1}{r}$, then

$$d_\infty(f^*, f_k) < r \quad \text{for all } k > N.$$

We conclude $f_k \to f^*$. $\square$

**Def** Let $x_n$ be a sequence in $(X, d)$. We say that $x_n$ is a <u>bounded</u> sequence if there exists some $r > 0$ such that

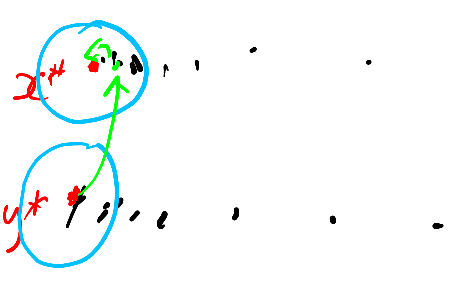
$$x_n \in B_r(x_0) \quad \text{for all } n.$$

Otherwise, we say $x_n$ is <u>unbounded</u>.

<u>Theorem</u> If $x_n$ is an unbounded sequence in $(X, d)$, then $x_n$ does not converge.

<u>Theorem</u> A sequence $x_n$ in $(X, d)$ can converge to at most one point.

<u>Proof</u> Suppose for the sake of contradiction that $x_n \to x^*$ and $x_n \to y^*$, where $x^* \neq y^*$.




Let $r = \frac{d(x^*, y^*)}{2}$.

Since $x_n \to x^*$, there exists some $N^{x^*}$ such that

$$d(x^*, x_n) < r \quad \text{for all } n \geq N^{x^*}.$$

Similarly, since $x_n \to y^*$, there exists some $N^{y^*}$ s.t.

$$d(y^*, x_n) < r \quad \text{for all } n \geq N^{y^*}.$$

Let $N = \max\{N^{x^*}, N^{y^*}\}$. Then

$$d(x^*, x_n) < r \text{ and } d(y^*, x_n) < r \text{ for all } n \geq N.$$

In particular, $d(x^*, x_N) < r$ and $d(y^*, x_N) < r$.

So $\underbrace{d(x^*, x_N) + d(y^*, x_N)}_{\text{a short-cut}} < 2r = d(x^*, y^*)$.

But this violates the triangle inequality ↯

**Def** We say that $y_n$ is a subsequence of $x_n$ if there exists a strictly increasing sequence $k_n \in \mathbb{N}$ (i.e. $k_{n+1} > k_n$) such that

$$y_n = x_{k_n}.$$

eg: if $x_n = \sqrt{n}$ and $y_n = \sqrt{2n}$,

$$x_n = 0, 1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}$$

$$y_n = 0, \sqrt{2}, \sqrt{4}, \sqrt{6}$$

then $y_n$ is a subsequence of $x_n$ $(k_n = 2n)$

<u>Theorem</u> If $x_n \to x^*$ and $y_n$ is a subsequence of $x_n$, then $y_n \to x^*$.

<u>Proof</u> The condition $x_n \to x^*$ means that for every radius $r > 0$, there exists some $N > 0$ such that

$$d(x^*, x_n) < r \quad \text{for all } n \geq N.$$

Since $y_n = x_{k_n}$ for some $k_n \geq n$, it follows

$$d(x^*, x_{k_n}) = d(x^*, y_n) < r \quad \text{for all } n \geq N.$$

So $y_n \to x^*$. $\square$

<u>Theorem</u> Consider $(\mathbb{R}, d_2)$. If $x_n \to x^*$ and $y_n \to y^*$ then

(i) $x_n y_n \to x^* y^*$,

(ii) $x_n + y_n \to x^* + y^*$, and

(iii) if $x_n \neq 0$ and $x^* \neq 0$, then $\frac{1}{x_n} \to \frac{1}{x^*}$.