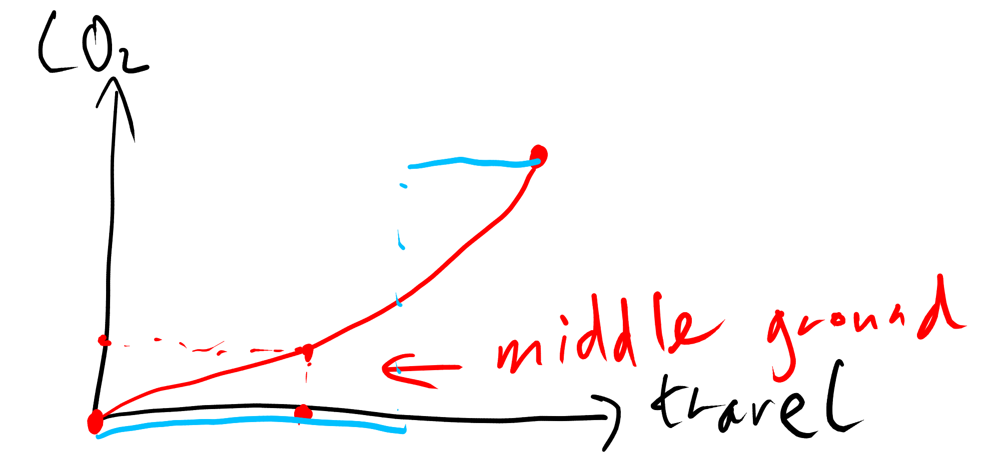
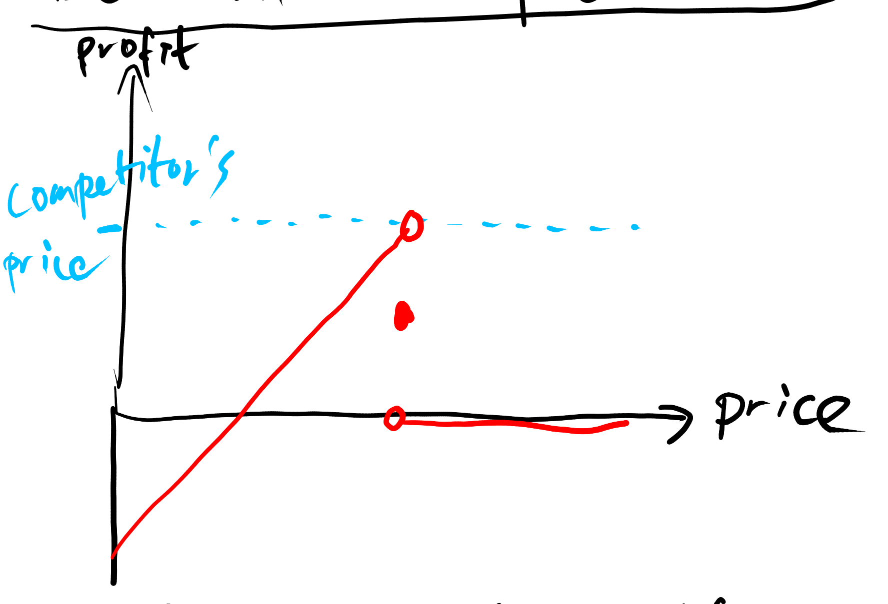
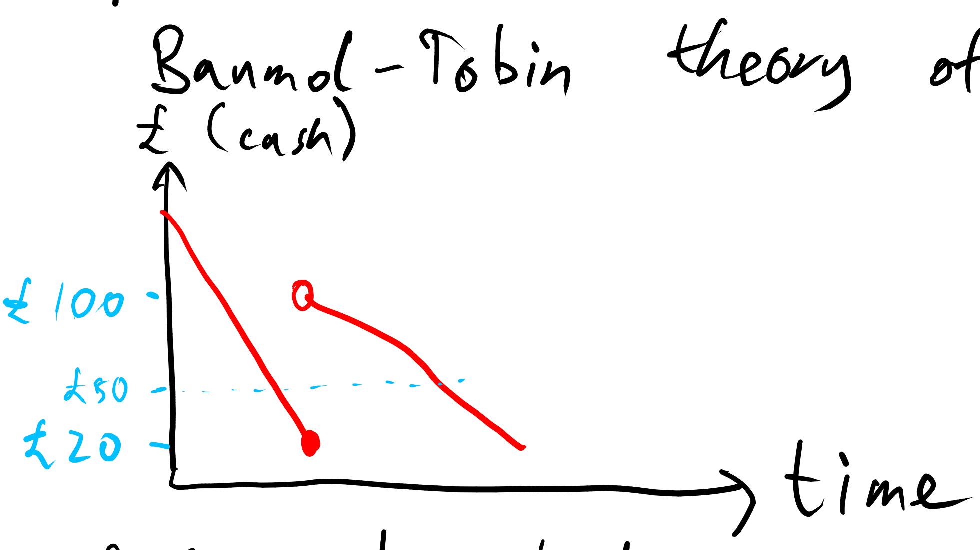


Bertrand competition



Baumol-Tobin theory of money demand

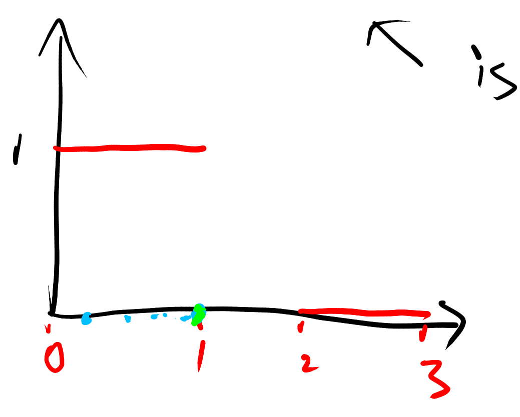


Def Consider metric spaces (X, d_X) and (Y, d_Y). We say that a function $f: X \to Y$ is continuous at x^* if for every convergent sequence $x_n \in X$ with $x_n \to x^*$, the sequence of images $f(x_n) \in Y$ converges with $f(x_n) \to f(x^*)$. We say f is continuous if f is continuous at all $x^* \in X$.

For example,

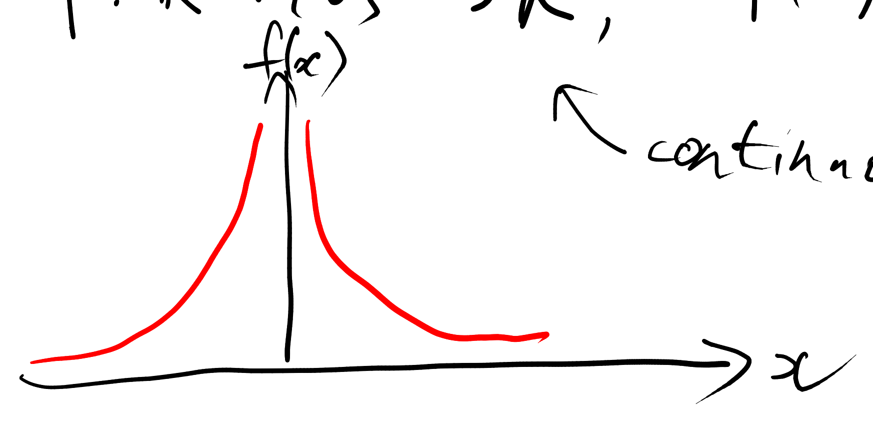
$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [2,3]. \end{cases}$$

is continuous!

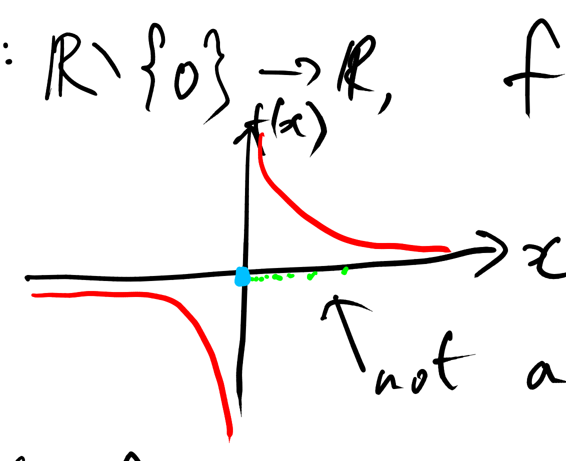


domain$(f) = [0,1] \cup [2,3]$.

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{|x|}$



$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is continuous!

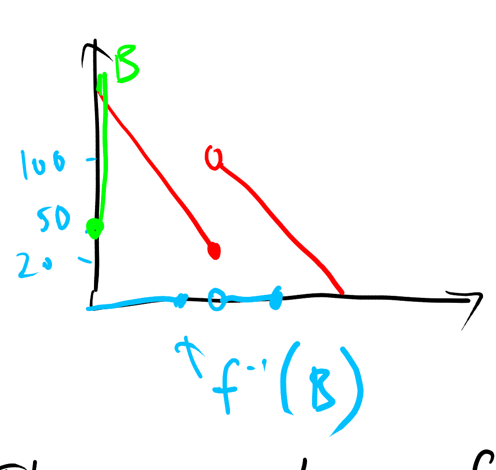


* If $f: X \to Y$ and d_X is the discrete metric, then f is continuous.

Def If $f: X \to Y$ and $A \subseteq X$ and $B \subseteq Y$, then

* the image of A is $f(A) = \{f(a) : a \in A\}$, and

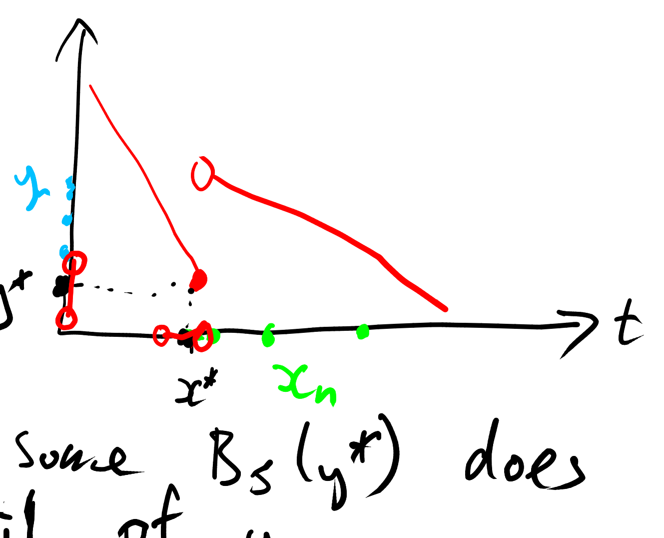
* the pre-image of B is $f^{-1}(B) = \{x \in X : f(x) \in B\}$.



Theorem Let $f: X \to Y$ be a function between (X, d_X) and (Y, d_Y). Pick any $x^* \in X$ and let $y^* = f(x^*)$. Then f is continuous at x^* if and only if for every open ball $B_\delta(y^*)$, there exists some open ball $B_r(x^*)$ such that $f(B_r(x^*)) \subseteq B_\delta(y^*)$.

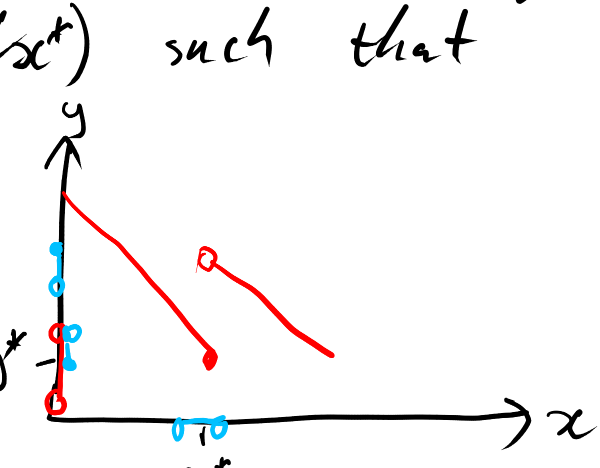
Proof We study the contrapositives of these statements. First, if f is open-set continuous, then f is sequentially continuous. Contrapositive: if f is sequentially discontinuous, then f is open-set discontinuous.

Suppose that for some sequence $x_n \to x^*$, we have $y_n = f(x_n) \not\to y^*$. We will an open ball $B_\delta(y^*)$ such that every open ball $f(B_r(x^*)) \nsubseteq B_\delta(y^*)$



Since $y_n \not\to y^*$, then some $B_\delta(y^*)$ does not contain any tail of y_n.
Since every open ball $B_r(x^*)$ contains a tail of x_n, it follows that every $f(B_r(x^*))$ contains a tail of y_n. Therefore, for every ball $B_r(x^*)$, we have $f(B_r(x^*)) \nsubseteq B_\delta(y^*)$.

Conversely, suppose f is open set discontinuous, i.e. for some ball $B_\delta(y^*)$, there is no ball $B_r(x^*)$ such that $f(B_r(x^*)) \subseteq B_\delta(y^*)$.
We will construct a sequence $x_n \to x^*$ such that $f(x_n) \not\to y^*$.



For every n, there exists some $x \in B_{\frac{1}{n}}(x^*)$ such that $f(x) \notin B_\delta(y^*)$. Let x_n be this point. Notice that $x_n \to x^*$ but $f(x_n) \not\to y^*$. □