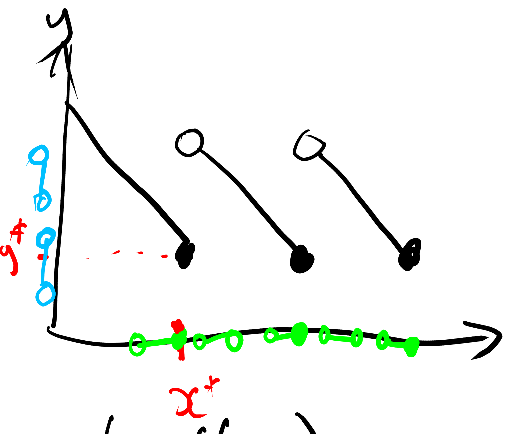
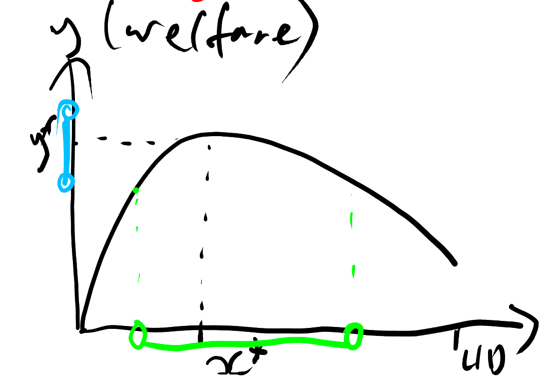
Theorem Let $f: X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if $f^{-1}(U)$ is an open set for all open sets $U \subseteq Y$.



— U

— $f^{-1}(U)$ is not open since $x^* \in f^{-1}(U)$ but x^* is not an interior point.



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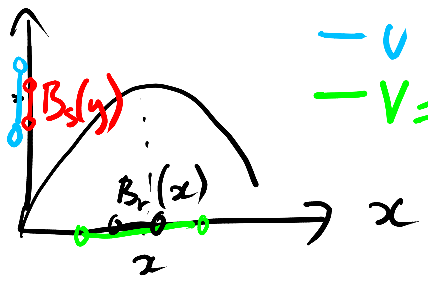
— $f^{-1}(U)$ is an open set

Proof Suppose f is continuous. Pick any open set $U \subseteq Y$, and let $V = f^{-1}(U)$. We need to prove that V is an open set. To this end, pick any $x \in V$. It suffices to show that x is an interior point of V. Let $y = f(x)$. Since U is open and $y \in U$ there is some ball $B_s(y) \subseteq U$.



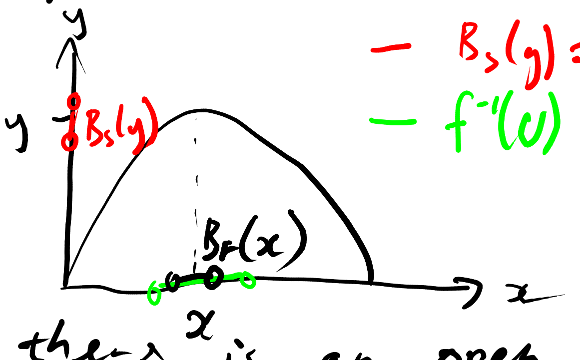
— U

— $V = f^{-1}(U)$

By the open ball continuity theorem, there is some ball $B_r(x)$ such that $f(B_r(x)) \subseteq B_s(y) \subseteq U$. It follows that $B_r(x) \subseteq f^{-1}(\underbrace{f(B_r(x))}_{\subseteq B_s(y)}) \subseteq f^{-1}(U)$.

We conclude that x is an interior point of V.

Conversely, suppose that for all open sets $U \subseteq Y$, the set $f^{-1}(U)$ is open. We will show that f is continuous at every $x \in X$. Pick any $x \in X$ and let $y = f(x)$, and pick any open ball $U = B_s(y)$.



— $B_s(y) = U$

— $f^{-1}(U)$

Since U is an open set, $f^{-1}(U)$ is an open set. Since $x \in f^{-1}(U)$, there is an open ball $B_r(x) \subseteq f^{-1}(U)$. This implies that $f(B_r(x)) \subseteq U = B_s(y)$. So the open ball continuity theorem implies f is continuous. $\square$