$$V(a) = \max_b \; v(a,b)$$

$$\text{s.t. } w(a,b) \geq 0.$$

What is $V'(a)$?

Lagrangian:

$$L(a,b,\lambda) = v(a,b) + \lambda w(a,b).$$

Recall FOC b: $\left.\frac{\partial L(a,b,\lambda)}{\partial b}\right|_{\substack{b=b(a)\\ \lambda=\lambda(a)}} = 0$

Expanding:

$$\left[\frac{\partial v(a,b)}{\partial b} + \lambda \frac{\partial w(a,b)}{\partial b}\right]_{\substack{b=b(a)\\ \lambda=\lambda(a)}} = 0.$$

Constrained envelope theorem

If V, v, w, b, λ are differentiable (continuously) and $\frac{\partial w}{\partial b} \neq 0$ and if the constraint binds ($w(a,b)=0$) then

$$V'(a) = \left.\frac{\partial L(a,b,\lambda)}{\partial a}\right|_{\substack{b=b(a)\\ \lambda=\lambda(a)}}$$

$$= \left[\frac{\partial v(a,b)}{\partial a} + \lambda \frac{\partial w(a,b)}{\partial a}\right]_{\substack{b=b(a)\\ \lambda=\lambda(a).}}$$

Proof Like the chain rule proof.

$$V(a) = L(a, b(a), \lambda(a))$$

$$= v(a, b(a)) + \lambda(a)\underbrace{w(a, b(a))}_{\text{assumed } =0}.$$

$$V'(a) = \Big[\underbrace{\frac{\partial L(a,b,\lambda)}{\partial a}}_{\text{direct effect}} + \underbrace{\cancel{\frac{\partial L(a,b,\lambda)}{\partial b}}}_{=0 \leftarrow \text{FOC}} \underbrace{b'(a)}_{\text{indirect effect}} + \underbrace{\cancel{\frac{\partial L(a,b,\lambda)}{\partial \lambda}}}_{=0 \text{ (constraint binds)}} \underbrace{\lambda'(a)}_{\text{constraint penalty}} \Big]_{\substack{b=b(a)\\ \lambda=\lambda(a).}}$$

$$= \left.\frac{\partial L(a,b,\lambda)}{\partial a}\right|_{\substack{b=b(a)\\ \lambda=\lambda(a)}}. \qquad \square$$

Recall the cost function:

$$c(y;w) = \min_{x \in \mathbb{R}^{N-1}_+} w.x$$

$$\text{s.t. } f(x) \geq y.$$

$$L(\underbrace{y,w}_{a};\underbrace{x}_{b};\lambda) = w.x - \lambda[f(x)-y]$$

The theorem says:

$$\frac{\partial c(y;w)}{\partial y} = \left.\frac{\partial L(y,w;x;\lambda)}{\partial y}\right|_{\substack{x=x(y,w)\\ \lambda=\lambda(y,w)}}$$

$$= \lambda \Big|_{\substack{x=x(y;w)\\ \lambda=\lambda(y;w)}}$$

$$= \lambda(y;w).$$

$$\frac{\partial c(y;w)}{\partial w_n} = \left.\frac{\partial L(y,w;x;\lambda)}{\partial w_n}\right|_{\substack{x=x(y,w)\\ \lambda=\lambda(y,w)}}$$

$$= x_n \Big|_{\substack{x=x(y,w)\\ \lambda=\lambda(y,w)}}$$

$$= x_n(y,w).$$

FOC: $\frac{\partial L(y,w;x;\lambda)}{\partial x_n}$

$$= w_n - \lambda \left.\frac{\partial f(x)}{\partial x_n}\right|_{\substack{x=x(y,w)\\ \lambda=\lambda(y,w)}}$$

$$= 0$$

$$\iff w_n = \lambda(y,w) \left.\frac{\partial f(x)}{\partial x_n}\right|_{x=x(y,w)}$$