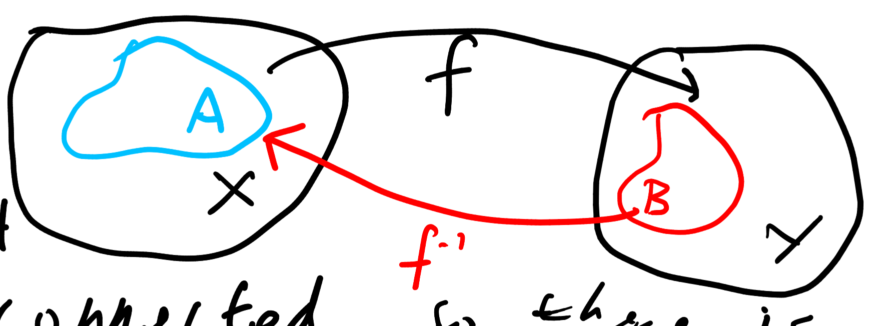
**Def** A metric space $(X, d)$ is connected if the only sets inside it that are both open and closed are $X$ and $\emptyset$. We say $A \subseteq X$ is connected if $(A, d)$ is a connected metric space.

Eg: $([0,1] \cup [2,3], d_2)$ is a disconnected metric space.

**Theorem** Consider two metric spaces $(X, d_X)$ and $(Y, d_Y)$. If $(X, d_X)$ is connected and $f : X \to Y$ is continuous and surjective, then $(Y, d_Y)$ is connected.

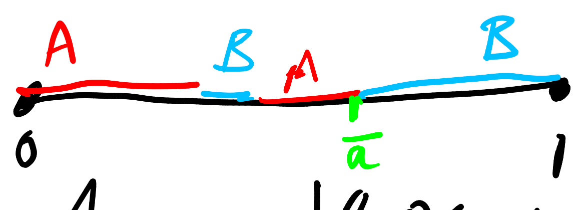



**Proof** Suppose for the sake of contradiction that $(Y, d_Y)$ is disconnected, so there is some non-trivial set $B \subseteq Y$ that is both open and closed. Let $A = f^{-1}(B)$.

Since $f$ is continuous, and $B$ is open, $A = f^{-1}(B)$ is open. Similarly $A$ is closed. Since $B$ is non-empty, $A \neq \emptyset$. Since $B \neq Y$, and $f$ is surjective, $A \neq X$. So $A$ is a non-trivial open & closed set. So $(X, d_X)$ is disconnected.

↯

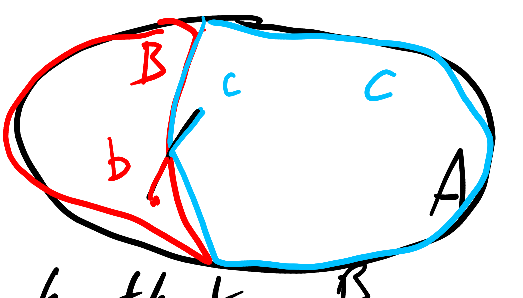
**Lemma** $([0,1], d_2)$ is connected.




**Proof** Suppose $A$ is both open & closed. Let $B = [0,1] \setminus A$. Assume WLOG $1 \in B$. Let $\bar{a} = \sup A$. Since $A$ is closed, $\bar{a} \in A$. Since $(\bar{a}, 1) \subseteq B$, and $B$ is closed, it follows that $\bar{a} \in B$.

↯

**Theorem** Consider any space $(\mathbb{R}^n, d_2)$. If $A \subseteq \mathbb{R}^n$ is convex, then $(A, d_2)$ is connected.

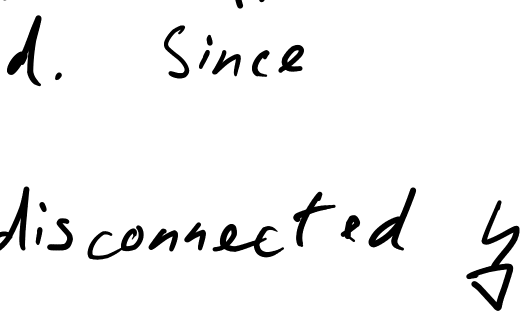



**Proof** Suppose for the sake of contradiction that $A$ is disconnected. Then there is some $B \subseteq A$ such that $B$ and $C = A \setminus B$ are open & closed in $(A, d_2)$. Pick any $b \in B$ and $c \in C$. Let $\bar{B} = B \cap [b, c]$ and $\bar{C} = C \cap [b, c]$. Both $\bar{B}$ and $\bar{C}$ are non-trivial open & closed sets in $([b, c], d_2)$. So $([b, c], d_2)$ is disconnected.

Consider the function $f : [0,1] \to [b, c]$ defined by $f(x) = xb + (1-x)c$. By the lemma, $([0,1], d_2)$ is connected. By the theorem, $([b, c], d_2)$ is connected.

↯

**Theorem** Consider $(\mathbb{R}, d_2)$. If $A$ is connected, then $A$ is convex.




**Proof** Suppose $A$ is not convex. Then there exists $a < b < c$ s.t. $a, c \in A$ but $b \notin A$.

Let $U = A \cap (-\infty, b)$ and $V = A \cap (b, \infty)$.

Next we show $U$ is closed inside $(A, d_2)$.

Let $u_n \in U$ be a convergent sequence with $u_n \to u^*$. Since $u_n \in U$, $u_n < b$. So $u^* \leq b$. But $b \notin A$, we deduce $u^* < b$. So $u^* \in U$, and $U$ is closed.

Similarly $V$ is closed. Since $U = A \setminus V$, $U$ is open.

Therefore $(A, d_2)$ is disconnected ↯

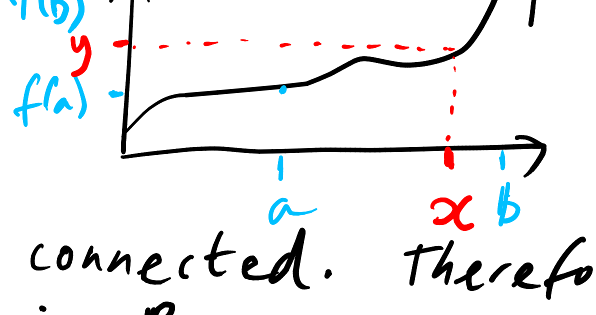
E.g.:

* $([0,1], d_2)$ is connected.
* $(\mathbb{R}^n, d_2)$ is connected
* Let $X = \{(x, y) : x^2 + y^2 = 1\}$. Then $(X, d_2)$ is connected.

  Let $f : [0, 2\pi] \to X$ defined by
  $$f(x) = (\cos x, \sin x).$$
  Since the domain is connected, and $f$ is continuous and surjective, the co-domain is connected.

**Theorem** (Intermediate Value Theorem)

Consider any continuous function
$$f : X \to \mathbb{R},$$
where $(X, d)$ is connected. If $a, b \in X$, then every $y \in [f(a), f(b)]$ has an inverse $x$ such that $y = f(x)$.




**Proof** Let $Y = f(X)$. Since $X$ is connected and $f$ is continuous, we know $(Y, d_2)$ is connected. Therefore, $Y$ is a convex set in $\mathbb{R}$.

Now, $f(a), f(b) \in Y$, so $y \in [f(a), f(b)]$ is also in $Y$. So $f^{-1}(y)$ is non-empty.

$\square$