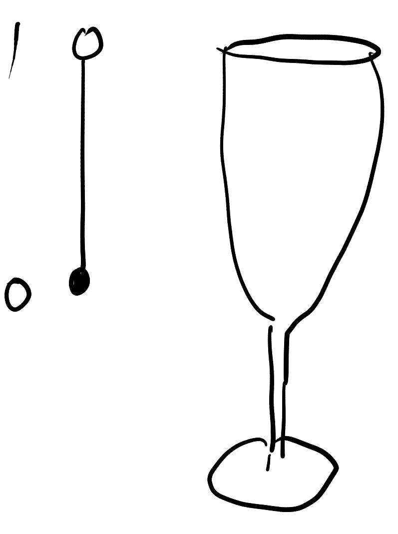
Complete spaces

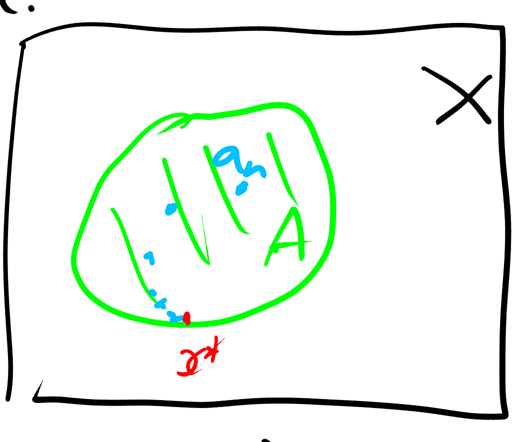
- $[0,1)$ is not closed in $(\mathbb{R}, d_2)$.
- $[0,1)$ is closed in $([0,1), d_2)$
- $([0,1), d_2)$ is not a complete metric space



<u>Def</u> We say a metric space (X,d) is <u>complete</u> if every Cauchy sequence $x_n \in X$ is convergent.

<u>Theorem</u> If (X,d) is a complete metric space and $A \subseteq X$ is a closed set, then (A,d) is a complete metric space.

<u>Proof</u> Pick any Cauchy sequence $a_n \in A$. Since $A \subseteq X$, $a_n \in X$.



Since (X,d) is complete, a_n is convergent inside (X,d). Let x^* be the limit of a_n in (X,d). Since A is closed and $a_n \to x^*$, we deduce $x^* \in A$. Therefore $a_n \to x^*$ in (A,d). So (A,d) is complete. □

Examples:

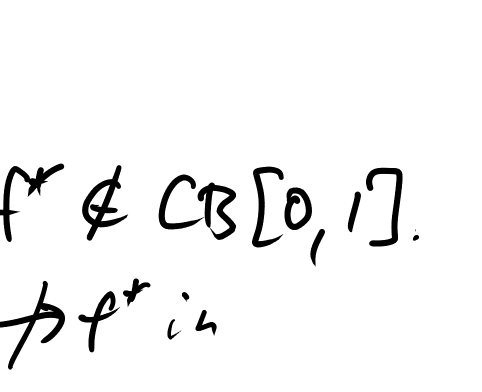
* $(\mathbb{R}, d_2)$ is complete
* $(\mathbb{R}^n, d_2)$ is complete
* $([0,1), d_2)$ is <u>not</u> complete
* $(\mathbb{Q}, d_2)$ is <u>not</u> complete. E.g.

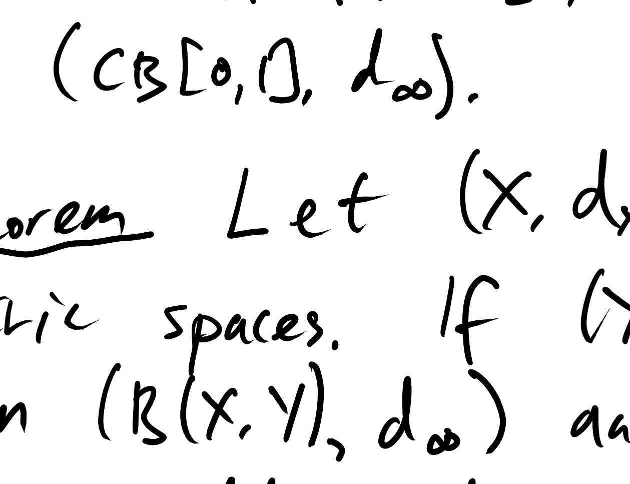
$x_1 = 3$
$x_2 = 3.1$
$x_3 = 3.14$
$x_4 = 3.141$

$x_n \to \pi$ inside $(\mathbb{R}, d_2)$

x_n does not converge inside $(\mathbb{Q}, d_2)$

* $(CB[0,1], d_1)$ where $d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. This is a metric space. but it is not complete

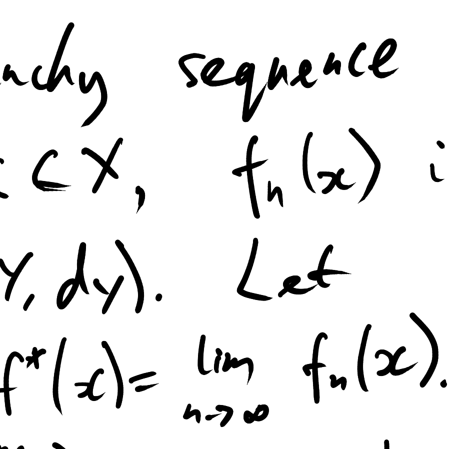




Note: $d_1(f_n, f^*) \to 0$, but $f^* \notin CB[0,1]$.

Note: $d_\infty(f_n, f^*) = \frac{1}{2}$, so $f_n \not\to f^*$ in $(CB[0,1], d_\infty)$.

<u>Theorem</u> Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then $(B(X,Y), d_\infty)$ and $(CB(X,Y), d_\infty)$ are complete metric spaces.



$f_n(x) = \frac{1}{n}$

$f^*(x) = 0$

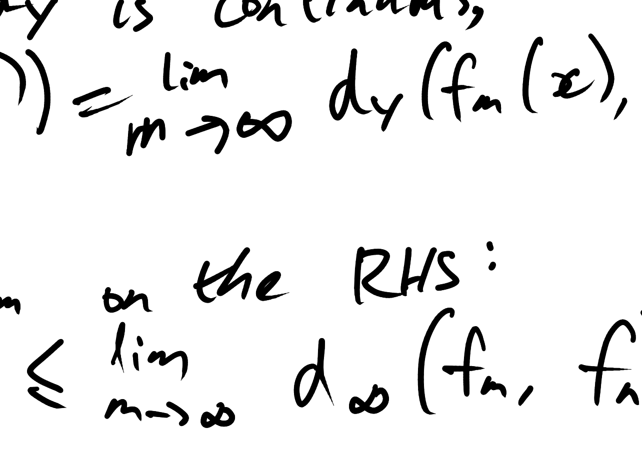
$d_\infty(f_n, f^*) \to 0$

but $f^* \notin B(X,Y)$

<u>Proof $(B(X,Y), d_\infty)$:</u> Let f_n be a Cauchy sequence in $(B(X,Y), d_\infty)$. Then for any $x \in X$, $f_n(x)$ is a Cauchy sequence inside (Y, d_Y). Let

$f^*(x) = \lim_{n\to\infty} f_n(x)$.

$f^*(x)$ exists because (Y, d_Y) is complete.



But we still need to prove that $f_n \to f^*$. Since d_Y is continuous,

$$d_Y(f^*(x), f_n(x)) = \lim_{m\to\infty} d_Y(f_m(x), f_n(x))$$

for all $x \in X$.

Take the supremum on the RHS:

$$d_Y(f^*(x), f_n(x)) \leq \lim_{m\to\infty} d_\infty(f_m, f_n).$$

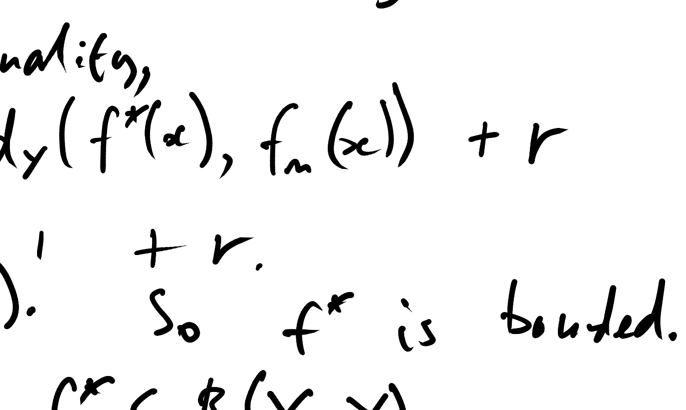
Since this is true for all $x \in X$,

$$d_\infty(f^*, f_n) \leq \lim_{m\to\infty} d_\infty(f_m, f_n).$$

Since f_n is a Cauchy sequence, the right side $\to 0$. So $d_\infty(f^*, f_n) \to 0$.

We check that $f^* \in B(X,Y)$, i.e that f^* is bounded.

Since $d_\infty(f_n, f^*) \to 0$, there exists some M such that $d_\infty(f_M, f^*) < 1$.



Since $f_M \in B(X,Y)$, there exists some ball $B_r(y)$ s.t. $f_M(X) \subseteq B_r(y)$.

By the triangle inequality,

$$d_Y(f^*(x), y) \leq d_Y(f^*(x), f_M(x)) + r < 1 + r.$$

So $f^*(X) \subseteq B_{1+r}(y)$. So f^* is bounded.

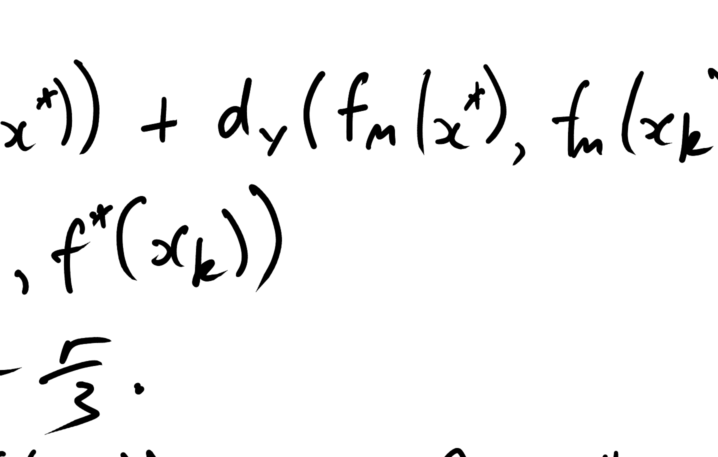
We conclude that $f^* \in B(X,Y)$, so $f_n \to f^*$, and hence $(B(X,Y), d_\infty)$ is complete.

<u>$(CB(X,Y), d_\infty)$ is complete:</u> We will prove $CB(X,Y)$ is a closed subset of $(B(X,Y), d_\infty)$. The previous then implies that $(CB(X,Y), d_\infty)$ is complete.

Let $f_n \to f^*$ inside $(B(X,Y), d_\infty)$, and suppose $f_n \in CB(X,Y)$. We want to prove f^* is continuous

Pick any $x_k \to x^*$ in (X, d_X). We want to prove $f^*(x_k) \to f^*(x^*)$. Pick any $r > 0$. Since $f_n \to f^*$, there is some N such that $d_\infty(f_N, f^*) < \frac{r}{3}$

Since f_M is continuous, there is some K such that $d_Y(f_N(x_k), f_N(x^*)) < \frac{r}{3}$ for all $k > K$.



Then for all $k > K$,

$$d_Y(f^*(x^*), f^*(x_k)) \leq d_Y(f^*(x^*), f_M(x^*)) + d_Y(f_M(x^*), f_M(x_k)) + d_Y(f_M(x_k), f^*(x_k)) < \frac{r}{3} + \frac{r}{3} + \frac{r}{3}.$$

So $d_Y(f^*(x^*), f^*(x_k)) < r$ for all $k > K$.

So $f(x_k) \to f^*(x^*)$ and f^* is continuous. So $CB(X,Y)$ is a closed set, and $(CB(X,Y), d_\infty)$ is complete. □