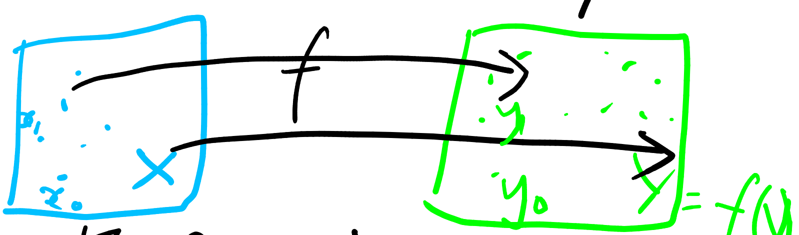
Def Let A be a subset of a metric space (X, d). We say A is <u>compact</u> if every sequence $x_n \in A$ has a convergent subsequence $y_n \to y^*$ such that $y^* \in A$. We say (X,d) is a <u>compact metric space</u> if X is a compact set.

Theorem Suppose (X,d) is a compact metric space. If $K \subseteq X$ is closed, then K is a compact set.

Proof Let $x_n \in K$. We want to prove that x_n has a convergent subsequence whose limit lies in K. Since (X,d) is compact, x_n has a convergent subsequence $y_n \to y^*$ where $y^* \in K$. Since K is closed, and $y_n \to y^*$, we conclude $y^* \in K$. So K is compact. $\square$

Theorem Suppose $f: X \to Y$ is a continuous function from (X, d_X) to (Y, d_Y). If (X, d_X) is compact and $Y = f(X)$, then (Y, d_Y) is compact.

Proof Let $y_n \in Y$ be any sequence in Y.



Since $Y = f(X)$, there exists a sequence $x_n \in X$ s.t. $y_n = f(x_n)$. Since (X, d_X) is compact, x_n has a convergent subsequence with indices n_k. So since f is continuous, $f(x_{n_k})$ is a convergent sequence, and a subsequence of y_n. So y_n has a convergent subsequence and (Y, d_Y) is compact. $\square$

Def A set is <u>bounded</u> if it's contained inside some open ball.

Theorem (Bolzano-Weierstass). Let A be a set in $(\mathbb{R}^n, d_2)$. Then A is compact iff A is closed & bounded.