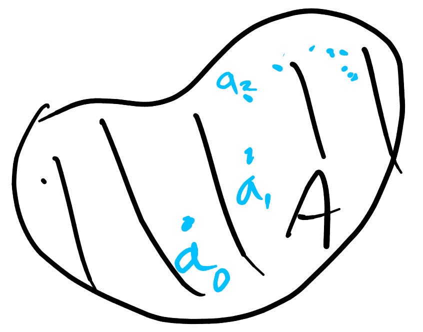
Def Suppose A is a set inside (X, d). We say A is <u>closed</u> if there is no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$.



For example:

* $[0,1]$ inside $(\mathbb{R}, d_2)$ is closed.
  e.g. $a_n = \frac{1}{n} \to 0 \in [0,1]$.
* X inside (X,d) is closed.
* $\emptyset$ inside (X,d) is closed.
* $(0,1)$ is closed inside $((0,1), d_2)$ but not inside $(\mathbb{R}, d_2)$.

Theorem Suppose A is a set inside (X, d). Then A is closed if and only if A contains its boundary, i.e. $\partial A \subseteq A$.

Proof <u>First, we show A is closed $\Rightarrow \partial A \subseteq A$</u>

If $x \in \partial A$, we want to show $x \in A$. If $x \in \partial A$, then there exists a sequence $a_n \in A$ s.t. $a_n \to x$. Since A is closed, $x \in A$, as required.

<u>Second, $\partial A \subseteq A \Rightarrow A$ is closed:</u>

Pick any convergent sequence $a_n \overset{\in A}{\to} x$. We want to prove $x \in A$. Assume for the sake of contradiction that $x \notin A$.

Then the sequence $b_n = x$ satisfies $b_n \notin A$ and $b_n \to x$. So a_n and b_n are appropriate sequences for establishing that $x \in \partial A$. But we assumed that $\partial A \subseteq A$, so $x \in A$. Contradiction ↯

Def Let A be a set in (X,d). The <u>closure</u> of A is

$$cl(A) = \{x^* \in X : \text{there exists a sequence } a_n \in A \text{ s.t. } a_n \to x^*\}$$

limit point