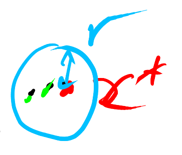
Def Let (X,d) be a metric space. A sequence $x_n \in X$ is a **Cauchy sequence** if for every radius $r>0$, there exists a number N such that
$$d(x_n, x_m) < r \quad \text{for all } n,m > N.$$

Theorem If $x_n \in X$ is convergent, then x_n is a Cauchy sequence.

Proof Suppose $x_n \to x^*$. Fix any radius $r>0$. By the def of convergence, there exists some N s.t.



$$d(x_n, x^*) < \tfrac{r}{2} \quad \text{for all } n > N.$$
By the triangle inequality,
$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x^*) + d(x^*, x_m) \\ &< \tfrac{r}{2} + \tfrac{r}{2} \\ &= r \quad \text{for all } n,m > N. \end{aligned}$$
We conclude that x_n is a Cauchy sequence. □

What about the converse?

No: $x_n = \frac{1}{n}$ inside $(\mathbb{R}_{++}, d_2)$ is not convergent. "Wants" to converge to 0, but $0 \notin \mathbb{R}_{++}$.

Theorem If $x_n \in X$ is a Cauchy sequence and $y_n \to y^*$ is a convergent subsequence of x_n, then $x_n \to y^*$.

Proof Pick any $r>0$. Since x_n is a Cauchy sequence, there is an $N>0$ s.t.
$$d(x_n, x_m) < \frac{r}{2} \quad \text{for all } n, m > N.$$
Since y_n is convergent, there exists some $k > N$ such that
$$d(y_k, y^*) < \tfrac{r}{2}.$$
By the triangle inequality,
$$\begin{aligned} d(x_n, y^*) &\le d(x_n, \underbrace{y_k}_{\text{an } x_m}) + d(y_k, y^*) \\ &< \tfrac{r}{2} + \tfrac{r}{2} \\ &= r. \end{aligned}$$
So $x_n \to y^*$. □

Theorem If $x_n \in X$ is a Cauchy sequence, then x_n is bounded.

Proof Since x_n is Cauchy, there is some N s.t. $d(x_n, x_m) < 1$ for all $n,m > N$.

Let $r = \max\{d(x_0, x_1), d(x_0, x_2), \dots d(x_0, x_{N-1}), 1 + d(x_0, x_N)\}$.

Then $x_n \in B_r(x_0)$. □

Theorem. If x_n is a Cauchy sequence and y_n is a subsequence of x_n, then y_n is a Cauchy sequence.