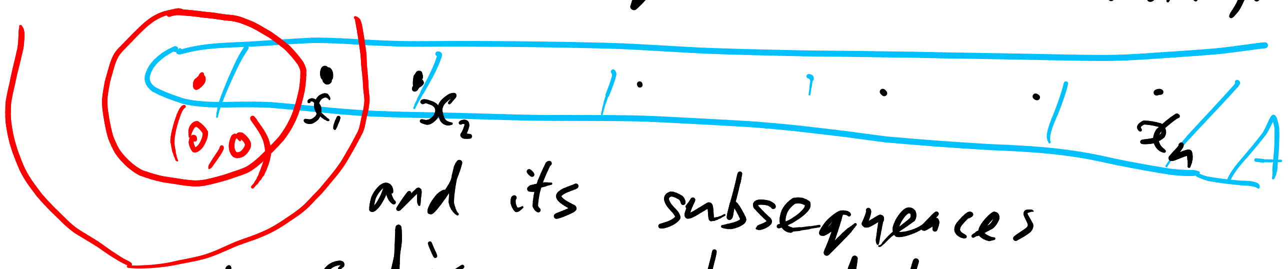
Theorem (Bolzano-Weierstrass)

Let A be a set in $(\mathbb{R}^n, d_2)$. Then A is compact iff A is closed and bounded.

Proof compact $\Rightarrow$ closed & bounded:

bounded: If A were unbounded, then no ball $B_r(0)$ would contain all of A.
$\uparrow \in \mathbb{R}^n$
So we can pick a sequence $x_n \in A \setminus B_n(0)$.



and its subsequences

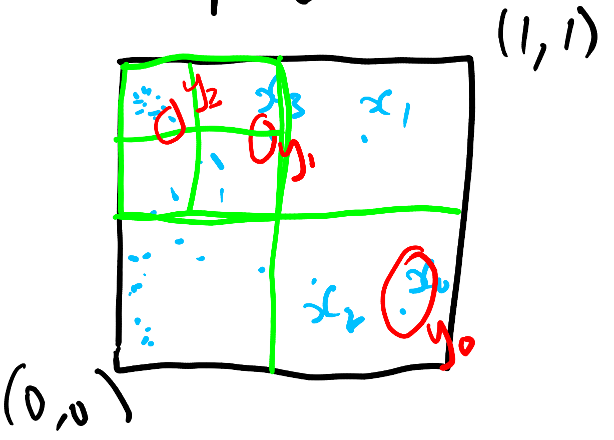
Notice that x_n is an unbounded sequence, and hence non-convergent. So x_n has no convergent subsequence, contradicting the assumption that A is compact.

closed: Suppose $x_n \in A$ s.t. $x_n \to x^*$. Since A is compact, some subsequence $y_n \to y^*$ and $y^* \in A$. Since sequences have only one limit, $y_n \to x^*$ and $x^* = y^*$ and $x^* \in A$. $\leftarrow$ any metric space

closed & bounded $\Rightarrow$ compact $\leftarrow$ $(\mathbb{R}^n, d_2)$ only

Plan: we will prove $([0,1]^n, d_2)$ is compact. This implies $([-r,r]^n, d_2)$ is compact since we can find a continuous function $f: [0,1]^n \to [-r,r]^n$. Since we can fit $A \subseteq [-r,r]^n$ (since A is bounded), and A is closed, we deduce A is compact.

So let's prove that $([0,1]^n, d_2)$ is compact. Draw for $n=2$.



Looking at the picture, we can find a subsequence of x_n called y_n s.t. $y_n, y_{n+1}, \dots$ is contained in a square of length $(\frac{1}{2})^n$.

So y_n is a Cauchy sequence. Since $(\mathbb{R}^n, d_2)$ is complete, we deduce y_n is convergent. $\square$

So these sets are compact:

* $[0,1]$ in $(\mathbb{R}, d_2)$
* $[0,1]^2$ in $(\mathbb{R}^2, d_2)$
* $[0,1] \cup [2,3]$ in $(\mathbb{R}, d_2)$.

Not compact:

* $(0,1)$ in $(\mathbb{R}, d_2)$
* $((0,1), d_2)$ $\leftarrow$ is closed & bounded in $((0,1), d_2)$ but not closed in $(\mathbb{R}, d_2)$
* $[0, \infty)$ in $(\mathbb{R}, d_2)$.