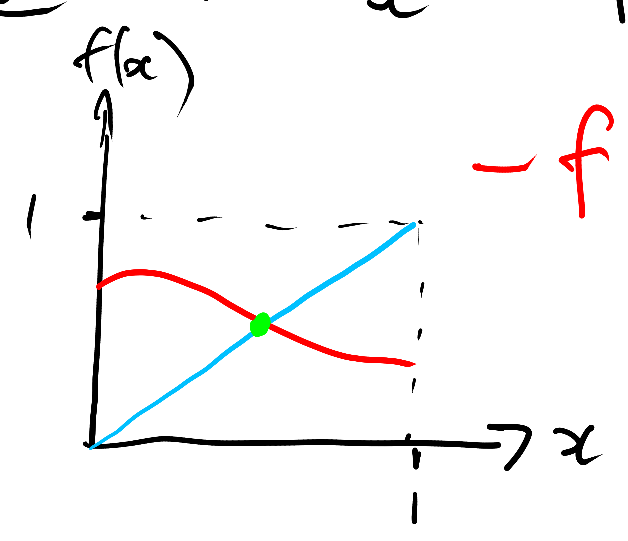
Fixed points

Def A function f is called a *self-map* if $f: X \to X$, i.e. the domain is the same as the codomain.

Def Let $f: X \to X$ be a self-map. A point $x^* \in X$ is a *fixed point* if $x^* = f(x^*)$.



Def Let (X, d_X) and (Y, d_Y) be metric spaces, and let $a > 0$. A function $f: X \to Y$ is *Lipschitz continuous* of degree a if for every $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq a\, d(x, x').$$

Def Let (X, d) be a metric space. We say a self-map $f: X \to X$ is a *contraction of degree $a<1$* if for all $x, x' \in X$,

$$d(f(x), f(x')) \leq a\, d(x, x').$$

Banach's fixed point theorem Let (X, d) be a complete metric space. If $f: X \to X$ is a contraction of degree a, then

(i) f has a unique fixed point x^*.

(ii) Given any $x_0 \in X$, the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*.

(iii) $d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1)$.

Proof *Uniqueness* Suppose x^* and x^{**} are both fixed points of f. As fixed points, $d(f(x^*), f(x^{**})) = d(x^*, x^{**})$.

But since f is a contraction

$$d(f(x^*), f(x^{**})) \leq a\, d(x^*, x^{**}).$$

So $d(x^*, x^{**}) = 0$ and $x^* = x^{**}$.

Existence and convergence: We will prove x_n is a Cauchy sequence. Applying the contraction property n times gives

$$d(x_n, x_{n+m}) = d(f^n(x_0), f^n(x_m)) \leq a^n d(x_0, x_m).$$

Now,

$$\begin{aligned} d(x_0, x_m) &\overset{\text{triangle inequality } m \text{ times}}{\leq} d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{m-1}, x_m) \\ &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots \text{ forever} \\ &\leq d(x_0, x_1) + a^1 d(x_0, x_1) + a^2 d(x_0, x_1) + \cdots \\ &= d(x_0, x_1)\left[1 + a^1 + a^2 + a^3 + \cdots\right] \\ &= \frac{1}{1-a} d(x_0, x_1). \end{aligned}$$

These two properties imply that

$$d(x_n, x_{n+m}) \leq a^n d(x_0, x_m) \leq \frac{a^n}{1-a} d(x_0, x_1);$$

for all n, m.

So $d(x_j, x_k) \leq \frac{a^N}{1-a} d(x_0, x_1)$ for all $j, k > N$.

We conclude x_n is a Cauchy sequence.

Since x_n is a Cauchy sequence in a complete metric space, x_n is convergent. We write $x_n \to x^*$. Since f is continuous, $y_n = f(x_n) \to f(x^*)$. But $y_n = x_{n+1}$ is a subsequence of x_n. So $x_n \to f(x^*)$. As x_n converges to both x^* and $f(x^*)$ we conclude $x^* = f(x^*)$. So x^* is a fixed point of f.

Approximation bound:

$$\begin{aligned} d(x_n, x^*) &\overset{\text{since } x_m \to x^* \text{ and } d \text{ is cont.}}{=} \lim_{m \to \infty} d(x_n, x_m) \\ &\leq \lim_{n \to \infty} \frac{a^n}{1-a} d(x_0, x_1) \quad \text{from the Cauchy formula.} \\ &= \frac{a^n}{1-a} d(x_0, x_1) \end{aligned}$$

□