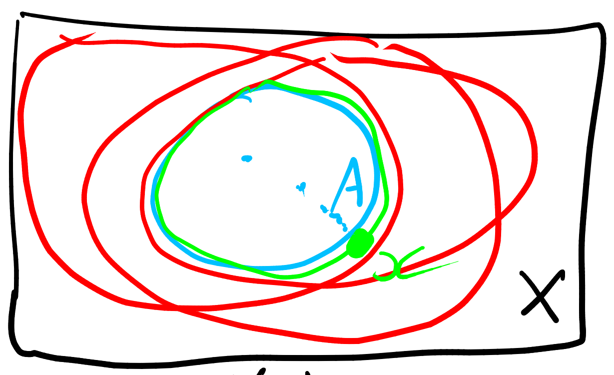
C.23

Let $\mathcal{C}$ be the set of closed sets that contain A. Let $I = \bigcap_{C \in \mathcal{C}} C$. Prove that

$$cl(A) = I.$$



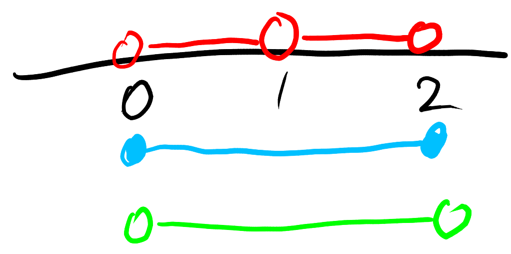
Closed sets that contain A.

① $I \subseteq cl(A)$. To see this, notice that $cl(A)$ is closed and $A \subseteq cl(A)$. So $cl(A) \in \mathcal{C}$. Since $I \subseteq C$ for all $C \in \mathcal{C}$, it follows that $I \subseteq cl(A)$.

② $cl(A) \subseteq I$. Pick any $x \in cl(A)$. We want to prove that $x \in I$. Since $x \in cl(A)$, there exists a sequence $a_n \in A$ such that $a_n \to x$. Pick any $C \in \mathcal{C}$. Since $A \subseteq C$, $a_n \in C$. Since C is closed, and $a_n \to x$, we deduce $x \in C$. Since the choice of C was arbitrary, $x \in C$ for all $C \in \mathcal{C}$. We conclude that $x \in I$. □

$\boxed{C.30}$

Find a counter-example to the false hypothesis $\text{interior}(cl(A)) = A$.



$A = (0,2) \setminus \{1\}$
$cl(A)$
$int(cl(A))$

If A is an open set, then $\text{interior}(cl(A)) \supseteq A$.

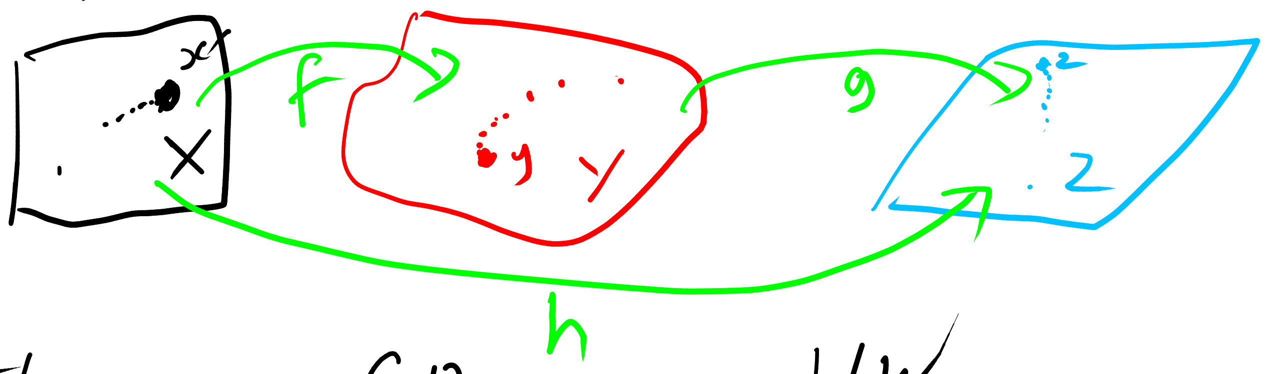
$\supseteq A$

That made me think, what kind of set A has a "much" bigger closure? $\Rightarrow$ holes

$\boxed{C.38}$

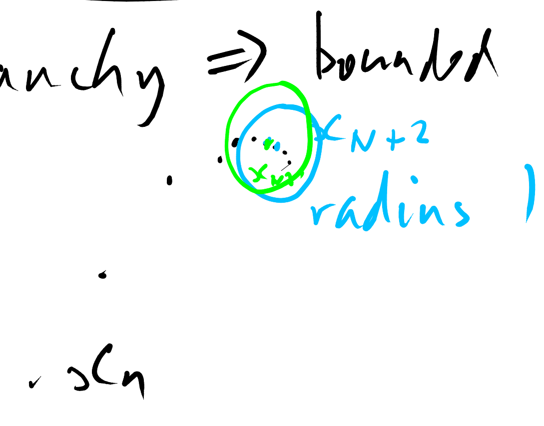
If $f: X \to Y$ and $g: Y \to Z$ are continuous, prove that $h(x) = g(f(x))$ is continuous.

Let $x_n \in X$ be a sequence with $x_n \to x^*$. Let $y_n = f(x_n)$ and $z_n = g(y_n) = h(x_n)$. Let $y^* = f(x^*)$ and $z^* = g(y^*)$. Since f is continuous, $y_n \to y^* = f(x^*)$. Since g is continuous, $z_n \to z^* = g(y^*) = g(f(x^*)) = h(x^*)$. So $h(x_n) \to h(x^*)$. So h is continuous at x^*. □



Theorem C.12

Cauchy $\Rightarrow$ bounded



HW

convergent $\Rightarrow$ bounded