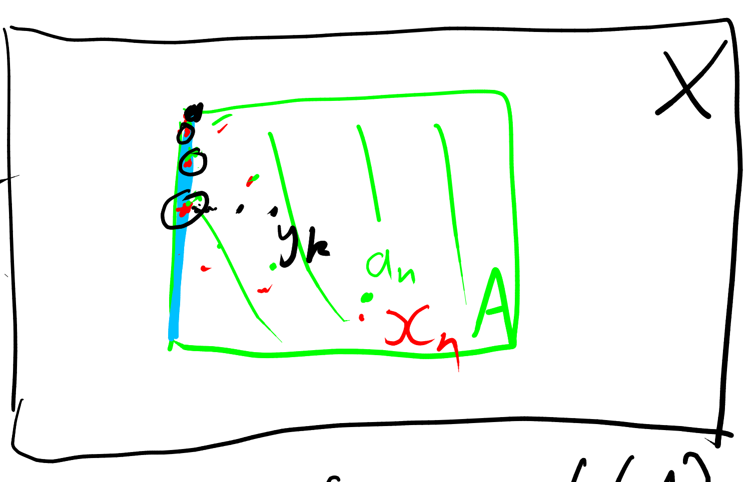
C.17 Let A be a set in the metric (X, d). Prove that the closure of A is a closed set.

— inside $cl(A)$ but outside A



Can't escape from $cl(A)$ by using points in A alone. (This is the def of $cl(A)$.) But what about escaping using points in $cl(A)$?

Proof strategy: let x_n be any sequence inside $cl(A)$, which converges to x^*. We will find another sequence $a_n \in A$ that also converges to x^*, and hence $x^* \in cl(A)$.

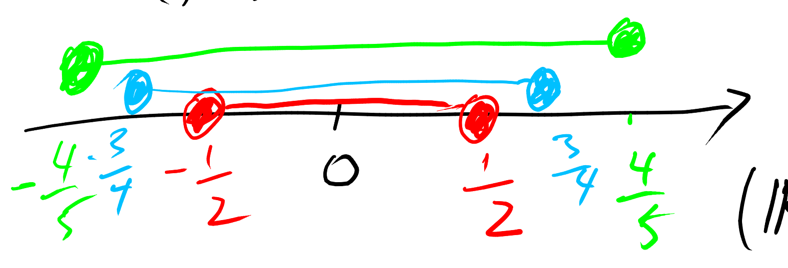
We will pick $a_n \in A$ to be within $B_{\frac{1}{n}}(x_n)$. This is possible because there some sequence $y_k \to x_n$ with $y_k \in A$, since $x_n \in cl(A)$.

Since $d(a_n, x_n) < \frac{1}{n}$ and $d(x_n, x^*) \to 0$, we have:

$$d(a_n, x^*) < d(a_n, x_n) + d(x_n, x^*) < \frac{1}{n} + d(x_n, x^*) \to 0.$$

We deduce that $a_n \to x^*$. So $x^* \in cl(A)$. $\square$

C.21



$(\mathbb{R}, d_2)$

— A_1
— A_2
— A_3

$A_n = \left[-1 + \frac{1}{n+1}, 1 - \frac{1}{n+1}\right]$

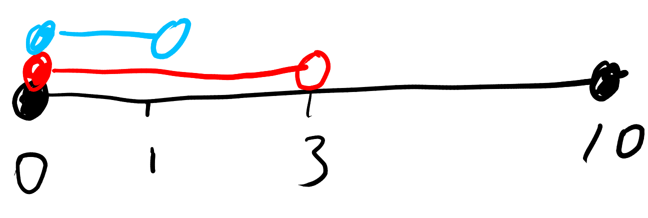
$$B = \bigcup_{n=1}^{\infty} A_n = \{a : a \in A_n, n \in \mathbb{N}\}$$

$-1 \in B$? No, because $-1 \notin A_n$ for all n.

Claim: $B = (-1, 1)$.

So A_n are closed but $\bigcup_{n=1}^{\infty} A_n$ is not closed.

C.25



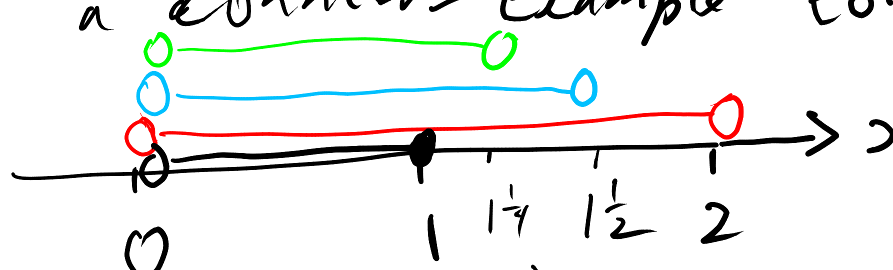
— X
— $B_2(1)$
— $B_1(0)$

$B_2(1) = [0, 3)$.

e.g. 0 is an interior point of $[0, 3)$ inside $([0, 10], d_2)$ because $B_1(0) = [0, 1) \subseteq [0, 3)$.

But $[0, 3)$ is not open inside the space $(\mathbb{R}, d_2)$.

C.27 Let $\{A_i\}_{i \in I}$ be a set of open sets, and let $B = \bigcap_{i \in I} A_i$. Find a counter-example to: B is open.

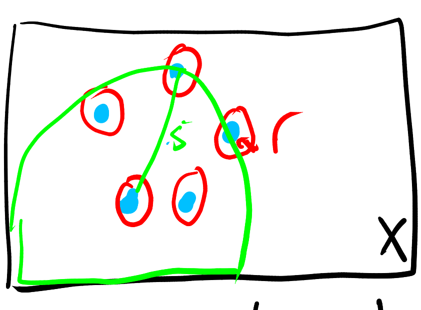


— A_0
— A_1
— A_2
— B

$$A_n = \left(0, 1 + \frac{1}{2^n}\right)$$

B is not open inside $(\mathbb{R}, d_2)$.

C.19



$\boxed{:.}$ A

We choose r to be small enough so that each ball $B_r(a)$ contains exactly one blue point.

s would be too big.

$\mathbb{R}_+ = \{x : x \geq 0\}$.

$\mathbb{R}^n = \{\underbrace{(x_1, \dots, x_n)}_{\text{tuple}} : x_i \in \mathbb{R}\}$

e.g. $1 \in \mathbb{R}_+$, $0 \in \mathbb{R}_+$ $-1 \notin \mathbb{R}_+$

$(0, 2) \in \mathbb{R}^2$, $3 \in \mathbb{R}^1$,

$(-3, 10, 20) \in \mathbb{R}^3$

$(2, 5) \notin \mathbb{R}^3$

$\mathbb{R}^3_+ = \{(x_1, \dots, x_3) : x_i \geq 0\}$

$(2, 3, 4) \in \mathbb{R}^3_+$, $(0, 0, 0) \in \mathbb{R}^3_+$

$(-1, 0, 0) \notin \mathbb{R}^3_+$.

$\mathbb{R}^{N-1}_{+}$ ← # inputs

← no negative quantities allowed

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \underbrace{\underbrace{(\mathbb{R} \times \mathbb{R})}_{\text{Cartesian product}} \times \mathbb{R}}_{\text{Cartesian product}}$$