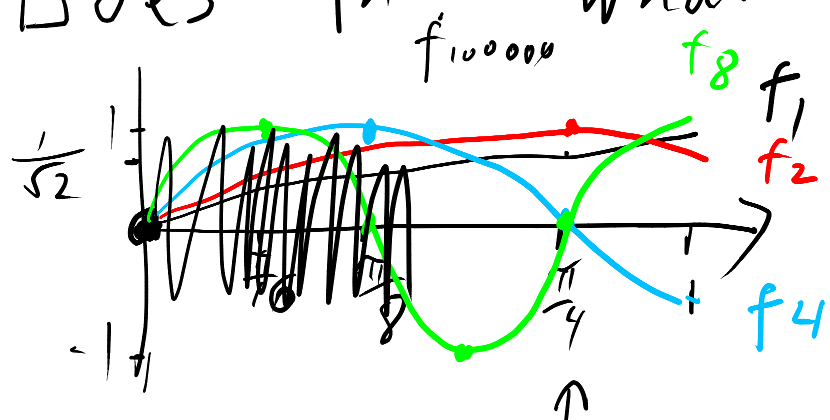
$\boxed{C7\,viii}$ Let $f_n(x) = \sin nx$.
Does f_n? What to?



claim If n is even, then $f_n(\frac{\pi}{4}) \in \{-1, 0, 1\}$

Proof Since n is even, $n = 2m$, so $f_n(\frac{\pi}{4}) = \sin(m\frac{\pi}{2})$ □

In fact, $f_{4n}(\frac{\pi}{4}) \in \{1, -1\}$, alternating between the two.

$d_\infty(f_n, f_m)$
$= \sup_{x \in [0,1]} |f_n(x) - f_m(x)|$
$\geq |f_n(\frac{\pi}{4}) - f_m(\frac{\pi}{4})|$

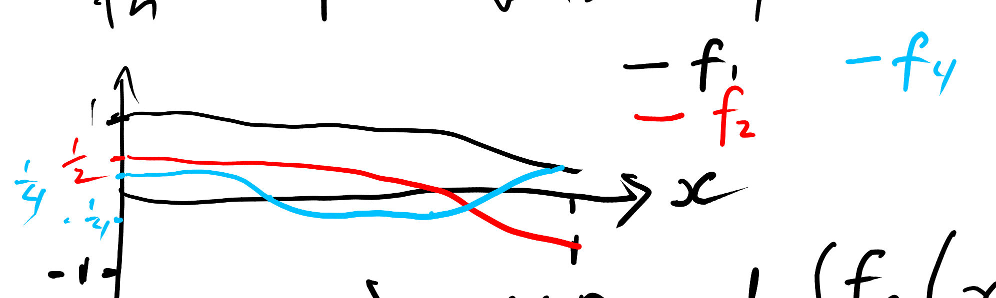
Now $f_{2n}(\frac{\pi}{4}) = 1, 0, -1, 0, 1, 0, -1, 0, \ldots$

$d_\infty(f_6, f_{10}) \geq 2$

$f_2(\frac{\pi}{4}) = 1$ $\quad f_6(\frac{\pi}{4}) = -1$ $\quad f_{10}(\frac{\pi}{4}) = 1$
$f_4(\frac{\pi}{4}) = 0$ $\quad f_8(\frac{\pi}{4}) = 0$

$\boxed{C.7\,ix}$ $f_n(x) = \frac{1}{n}\cos \pi n x$

A: $f_n \to f^*$ where $f^*(x) = 0$.



$d_\infty(f_n, f^*) = \sup_{x \in [0,1]} d_2(f_n(x), f^*(x))$
$= \sup_{x \in [0,1]} |\frac{1}{n}\cos \pi n x - 0|$
$= \frac{1}{n} \sup_{x \in [0,1]} |\cos \pi n x|$
$\leq \frac{1}{n} \cdot 1$

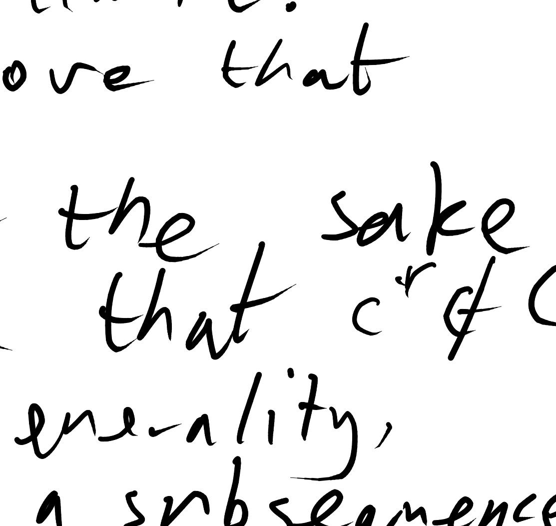
So $d_\infty(f_n, f^*) \to 0$, so $f_n \to f^*$.

Alternate finishing move:
Pick any $r > 0$.
If we set $N = \frac{1}{r}$
then $d_\infty(f_n, f^*) < r$ for all $n \geq N$.
So $f_n \to f^*$.

$\boxed{E8}$ If $g: Y \to X$ is surjective then
$$\max_{y \in Y} f(g(y)) = \max_{x \in X} f(x).$$

Let y^* maximise $f(g(y))$.
Let $x^* = g(y^*)$.
We want to prove x^* maximises f.
Let x^{**} be a maximum of f.
Then $f(x^*) \leq f(x^{**})$.
Since g is surjective, there exists y^{**} such that $g(y^{**}) = x^{**}$.
Now $f(g(y^*)) \geq f(g(y^{**}))$,
so $f(x^*) \geq f(x^{**})$.
We conclude $f(x^*) = f(x^{**})$.
So x^* also maximises f.

$\boxed{C.20}$ If A and B are closed sets inside (X, d), then
<u>$A \cup B$ is closed.</u>



Let $C = A \cup B$.
Pick any convergent sequence $c_n \in C$.
Let c^* be its limit.
We want to prove that $c^* \in C$.
Suppose for the sake of contradiction that $c^* \notin C$.
Without loss generality, assume c_n has a subsequence $a_n \in A$. Since a_n is a subsequence of c_n, $a_n \to c^*$.
Since A is closed, $c^* \in A$.
Since $A \subseteq C$, $c^* \in C$, a contradiction. ↯

<u>Theorem C.4(ii)</u> Consider $(\mathbb{R}, d_2)$.
If $x_n \to x^*$ and $y_n \to y^*$
then $z_n = x_n + y_n \longrightarrow z^* = x^* + y^*$.

<u>Proof</u>
Pick any $r > 0$.
$d_2(z_n, z^*) < r$?
$d_2(x_n + y_n, x^* + y^*) < r$
$= |(x_n + y_n) - (x^* + y^*)|$
$= |(x_n - x^*) + (y_n - y^*)|$
$\leq \underbrace{|x_n - x^*|}_{d_2(0,\, x_n - x^*)} + \underbrace{|y_n - y^*|}_{d_2(0,\, y_n - y^*)}$
$\geq d(x_n - x^*, y_n - y^*)$
$= |(x_n - x^*) - (y_n - y^*)|$

Since $x_n \to x^*$, there exists some N' s.t.
$d_2(x_n, x^*) < \frac{r}{2}$ for all $n > N'$.
Similarly for $y_n \to y^*$, there is an N''. Let $N = \max\{N', N''\}$

Then
$d_2(z_n, z^*)$
$\leq d_2(x_n, x^*) + d_2(y_n, y^*)$
$\leq \frac{r}{2} + \frac{r}{2}$
$\leq r$
for all $n > N$.
So $z_n \to z^*$.

<u>Clean up:</u>
Pick any $r > 0$. Since $x_n \to x^*$, there exists N_x s.t.
$d_2(x_n, x^*) < \frac{r}{2}$ for all $n > N_x$.
Similarly there is some N_y s.t.
$d_2(y_n, y^*) < \frac{r}{2}$ for all $n > N_y$.
Let $N = \max\{N_x, N_y\}$.
Then
$d_2(z_n, z^*)$
$= d_2(x_n + y_n, x^* + y^*)$
$= |(x_n + y_n) - (x^* + y^*)|$
$= |(x_n - x^*) + (y_n - y^*)|$
$\leq |x_n - x^*| + |y_n - y^*|$
$= d_2(x_n, x^*) + d_2(y_n, y^*)$
$< \frac{r}{2} + \frac{r}{2}$
$= r$
for all $n > N$.
So $z_n \to z^*$. □