

Assessment Guide and Practice Questions

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1 Assessment criteria

1.1 Assessment criteria for Microeconomics 1

My half of Microeconomics 1 is worth 50% of both the December and May exams. Your final mark is based on the maximum of your December and May exam marks.

My part of the class and degree exams have an identical format and marking scheme, approximately as follows (not counting the bonus questions). You will be rated as no/almost/ok/good/excellent (i.e. 0 to 4) on the following learning outcomes:

- Formulating a model. Excellence here means the absence of important mistakes.
- Breadth of technique: Walras law, dynamic programming, the envelope theorem, convex analysis, the first welfare theorem, etc. Excellence here usually means applying four techniques.
- Depth of technique: using a technique in an unusual way, combining several techniques to deduce something, or a clever piece of logic. For example, proving that there is at most one equilibrium in a particular model by combining the first welfare theorem with symmetry of all households. Obviously, depth requires at least some breadth, so this is correlated with the breadth learning outcome. Excellence here means that the assumptions, conclusions, and the logic from one to the other are clearly expressed.

There is no precise system for determining marks, but a linear regression reveals that marks will usually be close to $45.5 + 2.7m_1 + 3.8m_2 + 4.4m_3$, where m_i is the mark on learning outcome i on the 0 – 4 scale. This formula is less accurate at both extremes – all marks between 0 and 100 are possible.

Note that exams have become longer and more difficult in recent years to give multiple opportunities to show the third learning outcome. Thus, it has become easier to get a high mark.

1.2 Assessment criteria for Mathematical Microeconomics 1

You will sit two exams in December, and another two exams in May, all of which are three hours (12 hours total). Each pair of exams consists of the Microeconomics 1 exam and Part B of the Advanced Mathematical Economics exam. The Microeconomics 1 exams

are marked exactly the same way as for Microeconomics 1 students. These exams count for two thirds of your mark.

The mathematics exams count for the remaining third of your mark. Only your best exam (December or May) will be used. The mathematics exam will be marked against four criteria:

- Fundamentals (48 points). If you pass any other criterion below, you will automatically get all of these points.
- Definitions (10 points). Full marks will be awarded if you can reproduce four mathematical definitions.
- Reformulation (10 points). Writing a proof usually involves restating the question in a form that is convenient for writing a proof. For example, you might need to expand a definition, reformulate as a contrapositive, split up an if and only if into the two directions, and so on. Full marks will be awarded if you do useful reformulations for two questions.
- Deduction (32 points). This mark is determined by the quantity and quality of “snippets of logic.” I almost never give a mark in the 0-4 range. A mark in the range of 5-9 corresponds to being able to prove a simple theorem (with clear reasoning) or providing an example or counterexample. For example, proving that the interior and boundary of a set is disjoint would lead to a mark in this range. A mark in the range of 10-15 involves being able to prove two simple theorems. A mark in the range of 16-20 involves being able to prove one intermediate theorem (requiring many steps or integrating several ideas). A mark in the range of 21-32 involves writing a proof with a degree of mathematical creativity in combining ideas (such as 24.B.vii) or two intermediate theorems. It is ok to skip “easy” steps, just answer part of a question, and/or use theorems from lectures or homework. What matters is how you put it all together, and any logical manoeuvres or creative ideas you add in.

1.3 Assessment criteria for Advanced Mathematical Economics (undergraduate)

There are two versions of this course – undergraduate (ECNM10085) and postgraduate (ECNM11072). What follows here is the assessment for the undergraduate version. Homework is worth 10% of the mark. It is marked on effort only – if you attempt a majority of questions, you will score full marks. Homework can be submitted at the start of the lecture, or electronically via Learn. If you choose electronic submission, you can either scan, photograph, or type your homework. You might find Microsoft Lens convenient for this. You must submit at least 6 of the 9 problem sets. Otherwise, you will be penalised 2% for each additional problem set you missed. You will receive feedback on your homework during tutorials. I recommend that students ask each other for help, and also ask for help during Tutorials and also on Piazza.

One-semester visiting students can do an optional project about global warming. If the project mark is higher than the exam mark, then the final mark will be calculated

based on 10% homework + 45% project + 45% exam. If the project mark is lower than the exam mark, then the final mark will be calculated as 10% homework + 90% exam.

Exams are worth 90% of the mark. Full-year students can take both the December and May exams, whereas one-semester visiting students can only take the December exam. The better exam mark will be used to calculate the course mark. Exams are marked against the following criteria:

- Fundamentals (45 points). Students automatically get full marks on this criterion if they earn any marks on any of the other criteria. Otherwise, the mark on this criterion reflects the basic knowledge of mathematics and economics demonstrated by the student.
- Model formulation (10 points).
- Applying theorems to models (10 points). Full marks usually involves applying three techniques correctly.
- Proving mathematical theorems (35 points). This is by far the hardest learning outcome, and mostly corresponds to the questions in Part B. This mark is determined by the quantity and quality of “snippets of logic.” I almost never give a mark in the 0-4 range. A mark in the range of 5-9 corresponds to being able to prove a simple theorem (with clear reasoning) or providing an example or counterexample. For example, proving that the interior and boundary of a set is disjoint would lead to a mark in this range. A mark in the range of 10-15 involves being able to prove two simple theorems. A mark in the range of 16-20 involves being able to prove one intermediate theorem (requiring many steps or integrating several ideas). A mark in the range of 21-35 involves writing a proof with a degree of mathematical creativity in combining ideas (such as 24.B.vii) or two intermediate theorems.

For example, if you did all of your homework, made only minor mistakes in Part A, and were able to answer two of the easier Part B questions well, then your mark would be approximately, $10 + 0.9(45 + 9 + 9 + 15) = 80$.

I strongly recommend that students attempt Part A first, which is primarily about the model formulation and theorem application criteria. Part B is primarily about proving mathematical theorems.

I rarely set exam questions that appeared in the lecture notes – the class is about writing proofs, not memorising them. All material from the sections that I cover are examinable, unless I say it is not. For example, I said we skipped quasi-convexity in Appendix D. At the end of semester, I compile a comprehensive list of what is examinable. You can see the list from previous years by viewing “last years’ course materials”. The harder questions are designed to separate outstanding students (who deserve marks in the 90s) from excellent students. I would like it to be very transparent to students what they need to do to earn outstanding marks.

I believe all students have the potential to be outstanding, despite the fact that some students bring advantages with them at the beginning. I am doing my best to figure out how to bring out the best in all students. Students with low marks in the rest of their degree often achieve excellent results in my course. It would like this to happen even more often.

Questions 20, 21, 24, 27, 29, 31, 34 are specifically written for Advanced Mathematical Economics. All other questions are from Microeconomics 1, which covers different material. Many questions are based on material that was taught, but only in passing. These questions would be examinable as more advanced questions (and hence would attract a bigger reward if correctly answered.) These questions are *not* examinable, as they are based on ideas that were not covered in lectures at all:

- 1 (v), (vi), (vii).
- 2 –
- 3 (ii), (vii), (viii).
- 4 (iv), (vi), (vii).
- 5 (vi).
- 6 (ii), (vi), (vii).
- 7 (ii), (vi).
- 8 (ii), (iv).
- 9 (v).
- 10 (ii), (v), (vi).
- 11 (vi).
- 12 (iii), (v).
- 13 (ii), (vi), (vii).
- 14 (vi), (vii).
- 15 (ii), (v), (vi).
- 16 (iv), (vi).
- 17 (ii), (vi).
- 18 (ii), (vi), (vii).
- 19 (vi).
- 20 –
- 21 –
- 22 (iv), (v), (vi).
- 23 (ii), (v), (vi), (vii).
- 24 –

25 (ii), (iv), (v).

26 (iv), (v), (vi).

27 –

28 (ii), (vi).

29 –

30 (ii), (iii), (iv), (v).

31 –

32 (ii), (iv), (v).

33 (ii), (v), (vi).

34 –

1.4 Assessment criteria for Advanced Mathematical Economics (postgraduate)

The postgraduate version of Advanced Mathematical Economics is taken by CPD students and PhD students. It is different from Mathematical Microeconomics 1 (see above). The assessment is similar to the undergraduate version of Advanced Mathematical Economics, except:

- The weekly homework is not formally assessed.
- You must do the project, which is worth 20%.
- The exam(s) are only worth 80%.

As in the undergraduate version of the course, you can either enrol to take the course over one semester or the whole year.

2 Advice for Answering Exam Questions

2.1 Generic Advice

- There is no need to add extra complications into the model. For example, if the question does not mention time, then there is no need to put multiple time periods into the model.
- If you can't figure out the answer, don't pretend you know it. It's better to explain what you are confused about – a well written statement of confusion can illustrate that you know the material very well, and give you a very good mark.

- Even if you misformulate your model, this shouldn't stop you from answering subsequent parts. But if the model then seems inconsistent with the question (e.g. the question asks “show real wages are higher” when in your model, this is not true) then please do not try to prove the impossible. Instead, please either explain why the question is inconsistent, or if you're not sure, explain why you are stuck and can't complete your argument.
- Students often incorrectly identify the envelope formula as a first-order condition. It's not. First-order conditions are about optimal choices. If you are differentiating with respect to prices, you are not doing a first-order condition, because in competitive markets, nobody can choose prices.
- Students often confuse value functions and objective functions. For example, students often (mistakenly) write that a firm's first-order conditions with respect to the number of workers involves differentiating the profit function (rather than the firm's objective function) with respect to its labour input. But the profit function is a function of prices, not quantities, so it makes no sense to differentiate it with respect to a quantity.
- You can introduce assumptions at any point in the paper. For example, if you discover in part (iv) that you need to assume that the production function is concave, then you can write that assumption in your answer to part (iv). You do not need to revise your answer to part (i).
- In proofs and calculations, please write with complete grammatical sentences, including punctuation.
- In proofs, be careful to distinguish between “there exists” and “for all”.
- In proofs, be careful to distinguish between set membership (\in) and subsets (\subseteq).

2.2 A Checklist

A mark below 50% means something important was missing from your model. For example, you might have had two different markets with the same price, or a firm buying something (like a wholesale good) without using it in production. Here is a check-list of important ingredients of every economic model:

- Any notation is fine, but you must define it.
- When writing down the agents' optimisation problems, you should always write the choice variables under the *max*.
- In competitive models, agents only choose quantities, not prices.
- Every market has one (and only one) price. For example, labour markets have only one price if all types of labour are equally valued (by buyers and sellers). On the other hand, if workers have preferences over their profession, or firms value some workers above others, then these are separate markets and have separate prices.

- Every cost should also have a corresponding benefit (and vice versa). There are exceptions to this rule (e.g. inelastic labour supply), but think carefully about this.
- Every market should have a market-clearing condition. Thus, there are always an equal number of prices and market-clearing equations. It also means you need to define notation for both supply and demand. (In the sample solutions, I typically write firm decisions in upper case, and household decisions in lower case.)

2.3 Notation

Notation for partial derivatives: there are many common (correct) ways to write partial derivatives, including

$$\frac{\partial}{\partial x} f(x, y) \tag{1}$$

$$\frac{\partial f(x, y)}{\partial x} \tag{2}$$

$$f_x(x, y) \tag{3}$$

$$f_1(x, y) \tag{4}$$

$$D_x f(x, y) \tag{5}$$

$$D_1 f(x, y) \tag{6}$$

$$\nabla_x f(x, y) \tag{7}$$

$$\nabla_1 f(x, y). \tag{8}$$

Writing

$$f'_x(x, y) \tag{9}$$

is *not* standard, so I suggest you avoid it. (It is unambiguous though, so it wouldn't lose you marks in my exams.)

The notation $f'(x, y)$ or $Df(x, y)$ or $\nabla f(x, y)$ does *not* represent a partial derivative, but rather the *total* derivative, i.e. the vector (or matrix) of partial derivatives of f . Please don't write this if you mean a partial derivative.

3 Practice Questions

1: Micro 1, mock exam

Consider a pure-exchange economy in which all goods are produced from oil by home production over 2 time periods. Only oil is traded. There are two households and two oil deposit sites of size 1. The first site is owned by household A, and oil can be extracted from it at any rate over the 2 periods. The second site is owned by household B, but oil production is only possible in the second period. Both households have the same preferences, which are impatient discounted utility with the same per-period utility function which is strictly concave.

- (i) Define an equilibrium in this economy.

Answer: **Household A:** consumption c_1^A, c_2^A in periods 1 and 2, oil sales k_1^A, k_2^A , utility u , discount factor β , prices p_1, p_2 ,

$$\begin{aligned} & \max_{c_1^A, c_2^A, k_1^A, k_2^A \geq 0} u(c_1^A) + \beta u(c_2^A) \\ \text{s.t.} \quad & p_1 c_1^A + p_2 c_2^A = p_1 k_1^A + p_2 k_2^A \\ & k_1^A + k_2^A = 1. \end{aligned}$$

Household B.

$$\begin{aligned} & \max_{c_1^B, c_2^B \geq 0} u(c_1^B) + \beta u(c_2^B) \\ \text{s.t.} \quad & p_1 c_1^B + p_2 c_2^B = p_2 \cdot 1 \end{aligned}$$

Market clearing.

$$\begin{aligned} c_1^A + c_1^B &= k_1^A \\ c_2^A + c_2^B &= k_2^A + 1. \end{aligned}$$

Equilibrium. An equilibrium is a vector of quantities $c_1^{*A}, c_2^{*A}, c_1^{*B}, c_2^{*B}, k_1^{*A}, k_2^{*A}$ and prices p_1^*, p_2^* such that the quantities solve the households' problems above, and the markets clear.

- (ii) Write down the egalitarian social planner's problem (i.e. assuming that the social planner puts equal weight on the households.) What allocation would she choose?

Answer:

$$\begin{aligned} & \max_{c_1^A, c_2^A, c_1^B, c_2^B \geq 0} u(c_1^A) + u(c_1^B) + \beta u(c_2^A) + \beta u(c_2^B) \\ \text{s.t.} \quad & c_1^A + c_1^B \leq 1 \\ & c_1^A + c_1^B + c_2^A + c_2^B \leq 2. \end{aligned}$$

The social welfare function is strictly concave, so there is a unique optimal allocation. By inspection, the first-order conditions for the two households are identical,

so the solution gives both households the same consumption paths. Thus, the social planner's problem reduces to:

$$\begin{aligned} & \max_{c_1, c_2 \geq 0} 2u(c_1) + \beta 2u(c_2) \\ \text{s.t.} \quad & 2c_1 \leq 1 \\ & 2c_1 + 2c_2 \leq 2. \end{aligned}$$

Does the first constraint bind? To check, we will solve without it. The FOCs w.r.t. c_1 and c_2 without the constraint is:

$$2u'(c_1) = \lambda 2 \quad \text{and} \quad 2\beta u'(c_2) = \lambda 2,$$

where λ is the Lagrange multiplier on the second constraint. Hence, $u'(c_1) = \lambda < \lambda/\beta = u'(c_2)$, which means that $u'(c_1) < u'(c_2)$. Since u is concave, u' is decreasing, so $c_1 > c_2$. Thus, the constraint is violated if we drop it. We conclude that it binds, which means that $c_1 = 0.5$ and $c_2 = 0.5$.

Summary: because the social planner values both households equally, and the households have strictly concave utility, both households follow equal consumption paths. The households are impatient, so the social planner is tempted to give them more consumption in the first period than the second. However, this is infeasible, because there is not enough oil in the first period. Thus, the households have equal consumption across time.

(iii) In equilibrium, how do oil prices change over time?

Answer: The FOCs for households A are

$$u'(c_1^A) = \lambda^A p_1 \quad \text{and} \quad \beta u'(c_2^A) = \lambda^A p_2.$$

Dividing the top by the bottom gives

$$\frac{u'(c_1^A)}{u'(c_2^A)} = \beta \frac{p_1}{p_2},$$

or equivalently,

$$p_2 = \frac{u'(c_2^A)}{u'(c_1^A)} \beta p_1.$$

A similar procedure gives

$$p_2 = \frac{u'(c_2^B)}{u'(c_1^B)} \beta p_1.$$

Since aggregate consumption can not be bigger in period 1, one (and hence both) of the fractions must be less than one. Hence $p_1 > p_2$.

- (iv) In equilibrium, which household is better off? Explain.

Answer: Since prices are decreasing over time, the first household's endowment is worth more.

- (v) Suppose there is a bubble, in the sense that in the last period, oil prices are too high and there is excess supply of oil in the last period. What would happen in the first period? (Hint: Walras' law.)

Answer: Walras' law applies. (The version in class was only for pure-exchange economies; oil storage can easily be accommodated with home production.) Walras' law says that there must be excess demand in some other market. Since there are only two markets, it must be excess demand for oil in the first period.

- (vi) * Which assumptions above about the households' utility are relevant for Debreu's theorem about additively separable preferences? Which assumptions go beyond the conclusion of Debreu's theorem?

Answer: The question assumes the conclusion of Debreu's theorem (and more), that preferences are additively separable. Debreu requires there to be at least three time periods, and for preferences to be additively separable. The assumptions of discounted utility and impatience are additional assumptions made by the model.

- (vii) * What additional assumptions are needed to ensure existence of equilibria in this economy?

Answer: None. The utility functions are strictly quasi-concave, so the excess demand function is continuous. The choice spaces are convex and compact, so the proof of the existence theorem would not have any serious obstacles.

2: Micro 1, mock exam

The cashew tree is native to the Amazon forest in Brazil, its fruit is about the same size as an apple. The juice of the flesh of the fruit is popular in Brazil (along with açaí, acerola, guava, mango, papaya, and many others... but ignore those!) Each fruit has enough juice to fill a single cup. Each fruit also contains a single seed, which when toasted becomes the cashew nut which is popular all over the world.

- (i) The firm chooses how many cashew fruits to grow (which requires labour), and then sells the juice and nuts. Assume that no work is required to extract the juice and nuts – only growing requires labour. Write down the firm's profit function.

Answer: Notation: J juice, N nuts, L labour, $f(L)$ fruit production function, p^J, p^N, p^L prices

$$\pi(p^J, p^N, p^L) = \max_L (p^J + p^N)f(L) - Lp^L.$$

- (ii) Write down the firm's cost function. Hint: you will need two quantities in the state variable (as well as factor prices).

Answer: Notation: Y^J, Y^N are production targets,

$$C(Y^J, Y^N, p^L) = \min_L Lp^L$$

s.t. $f(L) \geq Y^J$ and $f(L) \geq Y^N$.

- (iii) There are several identical households that supply labour and consume cashews and cashew juice and hold equal shares in the cashew firm. Write down a general equilibrium model of the economy.

Answer: Focus on symmetric equilibria, in which all households make the same decisions.

Households: H households, $\Pi = \pi(p^J, p^N, p^L)$ aggregate profits,

$$\max_{c^J, c^N, l} u(c^J, c^N, 1 - l)$$

s.t. $p^J c^J + p^N c^N = p^L l + \Pi/H$.

Firms: As above.

Equilibrium: Prices (p^{*J}, p^{*N}, p^{*L}) and quantities (c^{*J}, c^{*N}, l^*) for households and (Y^{*J}, Y^{*N}, L^*) for the firm such that:

- the quantity decisions are optimal given prices (see above), and
- all markets clear:

$$Hc^{*J} = Y^{*J}$$

$$Hc^{*N} = Y^{*N}$$

$$Hl^* = L^*.$$

- (iv) Write down a utility function for the households consistent with the idea that households enjoy cashew nuts more than cashew juice. What can you say about equilibrium prices in this case?

Answer: For example, pick $u(c^J, c^N, r) = \log c^J + 2 \log c^N + r$, where r is relaxation time. The FOCs for c^J and c^N are

$$\frac{1}{c^J} = \lambda p^J$$

$$\frac{2}{c^N} = \lambda p^N.$$

From the firm's production technology and market clearing, we know that $c^J = c^N$ in all (symmetric) equilibria. Dividing the second FOC by the first and rearranging gives

$$p^N = 2p^J,$$

i.e. cashew nuts are twice as expensive in this example. Even though the social cost of producing nuts is the same as juice, the marginal social opportunity cost of consuming a nut is higher, because it deprives the other households of something more valuable.

- (v) Does the firm have increasing marginal cost in both products?

Answer: Yes, the cost function

$$C(Y^J, Y^N; w) = \min_{L \geq 0} wL$$

$$\text{s.t. } f(L) \geq \max \{Y^J, Y^N\}$$

is convex in the output targets. If the cheapest way to produce $Y = (Y^J, Y^N)$ is L units of labour, and to produce $\hat{Y} = (\hat{Y}^J, \hat{Y}^N)$ is \hat{L} units, then we just need to check that producing $aY + (1-a)\hat{Y}$ output requires at most $aL + (1-a)\hat{L}$ labour. To see this, $C(Y; w) = wL$ and $C(\hat{Y}; w) = w\hat{L}$, and if $aL + (1-a)\hat{L}$ meets the production targets $aY + (1-a)\hat{Y}$, then

$$C(aY + (1-a)\hat{Y}; w) \leq w[aL + (1-a)\hat{L}] = aC(Y; w) + (1-a)C(\hat{Y}; w).$$

Looking at the juice, we know that

$$f(L) \geq Y^J \tag{10}$$

$$f(\hat{L}) \geq \hat{Y}^J, \tag{11}$$

because L and \hat{L} labour generates at least these amounts of juice. By the concavity of the production function f , we know that $f(aL + (1-a)\hat{L}) \geq af(L) + (1-a)f(\hat{L})$. Thus, taking the convex combination of the equations (10) and (11) and combining with this convex inequality gives

$$f(aL + (1-a)\hat{L}) \geq aY^J + (1-a)\hat{Y}^J.$$

That is, the intermediate amount of labour produces at least the intermediate amount of juice. A similar line of reasoning applies to the nuts.

- (vi) Sketch a graph of the firm's marginal cost of producing cashew juice, holding fixed the number of cashew nuts being produced at 3.

3: Micro 1, December 2012

A farm produces food from labour. However, the farm does not have a distribution network, so it can not sell the food directly to the households. Rather, it must sell the food to a supermarket at a wholesale price, which then resells to households at a retail price. The supermarket buys food and labour, which it uses to resell the food. Some food might get wasted; more labour means less food gets wasted. All households are identical, and supply labour to both firms.

- (i) Formulate an economy by writing down the households' and firms' value functions, and the market clearing conditions. Focus attention on symmetric equilibria, i.e. in which all households make the same decisions. (Hint: you might find it helpful to consider the wholesale food a completely separate good. Don't forget profits.)

Answer: **Household.** p retail food price, w wage, c consumption, l labour, H number of households, $u(c, l)$ utility function, $\Pi = \Pi^F + \Pi^S$ firms' profits, value

$$v(p, w) = \max_{c, l} u(c, l)$$

$$\text{s.t. } pc = wl + \frac{\Pi}{H}.$$

Farm. D_F wholesale good produced, $D_F = f(L_F)$ production function, ϕ wholesale price, value

$$\pi^F(\phi, w) = \max_{L_F} \phi f(L_F) - wL_F.$$

Supermarket. D_S wholesale good purchased, C_S retail food sold, $C_S = g(L_S, D_S)$ production function, value

$$\pi^S(p, \phi, w) = \max_{L_S, D_S} pg(L_S, D_S) - \phi D_S - wL_S.$$

Equilibrium. A symmetric allocation consists of quantities for households (c^*, l^*) , the farm (D_F^*, L_F^*) , and the supermarket (C_S^*, D_S^*, L_S^*) . These choices, along with prices (p^*, ϕ^*, w^*) and profits (Π^{F^*}, Π^{S^*}) form an equilibrium if the

- choices solve the problems defined above,
- profits match: $\Pi^{S^*} = \pi^S(p^*, \phi^*, w^*)$ and $\Pi^{F^*} = \pi^F(\phi^*, w^*)$.
- food clears: $Hc^* = C_S^*$.
- wholesale clears: $D_S^* = D_F^*$.
- labour clears: $Hl^* = L_S^* + L_F^*$.

- (ii) Select a constraint which may be dropped by Walras' law.

Answer: Eg: "food clears."

- (iii) Suggest how an endogenous variable may be eliminated, since inflation of all prices by an equal factor does not affect decisions.

Answer: Eg: set $w^* = 1$.

- (iv) Show that the supermarket's profit function is convex. (Hint, you may use the following theorem from class: Suppose V is the upper envelope of convex functions, i.e. $V(a) = \max_b v(a, b)$ where $v(\cdot, b)$ is a convex function for each b . Then V is convex.)

Answer: To apply the theorem, the choice variable b corresponds to the quantities (D_S, L_S) , the state variable a corresponds to prices (p, ϕ, w) , and the function $v(a, b)$ corresponds to $pg(L_S, D_S) - \phi D_S - wL_S$, which is linear in prices. Since linear functions are convex, the theorem implies that the upper envelope, $\pi^S(p, \phi, w)$ is convex.

- (v) Show that the supermarket responds to a wholesale price increase by buying less.

Answer: By the envelope theorem,

$$\frac{\partial \pi^S(p, \phi, w)}{\partial \phi} = \frac{\partial}{\partial \phi} [pg(L_S, D_S) - \phi D_S - wL_S]_{L_S=L_S(p, \phi, w), D_S=D_S(p, \phi, w)} = -D_S(p, \phi, w).$$

Differentiating and multiplying by -1 on both sides gives

$$-\frac{\partial^2 \pi^S(p, \phi, w)}{\partial \phi^2} = \frac{\partial D_S(p, \phi, w)}{\partial \phi}.$$

Since π^S is convex, the left side is negative. Thus, the right side is negative, so the sales policy is decreasing in the wholesale price ϕ .

- (vi) There have been protests recently that the (equilibrium) retail price is much higher than the wholesale price, which the households feel is grossly unfair. They propose introducing a profit tax of 50% to be redistributed equally among households, a price markup ceiling of 10%, and a minimum wage increase of 20%. Would this policy make the households better off (under standard assumptions, like increasing utility functions)?

Answer: No. By the first welfare theorem, the original allocation was efficient. Thus, it is infeasible to make all households better off. In fact, since all households have the same budget constraint and utility function, they all attain the same equilibrium utility, so no household would be better off.

- (vii) * Prove that the supermarket's policy is continuous if its production function is strictly concave. You may assume that the supermarket only has space to accommodate a maximum number of workers and amount of food.

Answer: The strict concavity of the firm's objective implies that the optimal policy $\psi(P)$ as a function of the price vector P is unique. Now suppose for the sake of contradiction that a sequence of price vectors P_n converges to P^* , but that $\psi(P_n)$ does not converge to $\psi(P^*)$. Since the number of workers and food are limited, the choice space is compact, so we may assume without loss of generality that $\psi(P_n)$ converges to some point, (L, D) . But by continuity of the supermarket's objective, (L, D) and $\psi(P^*)$ give the same profit, which contradicts the uniqueness of the optimal policy.

- (viii) * To prove existence of equilibrium using Brouwer's fixed point theorem, it is important that the set of possible prices are compact. Explain why this is important, and how to accommodate this requirement.

Answer: One way to prove existence is to show that there is a fixed point of some price-adjustment function $\phi : P \mapsto P'$. Boundedness of the possible price set is important, as inflation might rule out fixed points (eg: $\phi(P) = P + (1, \dots, 1)$ has no fixed point.) Closedness is important to rule out a hole at a point that would have been the fixed point. It is straightforward to compactify the price set by normalising prices rescaling them to sum to 1. This is possible, because only relative prices matter – rescaling does not affect incentives.

4: Micro 1, December 2012

Sackman, Erickson, and Grant (1968) conducted an experiment on computer programmers, which they published in the Communications of the Association of Computing Machinery. They summarised their findings with the following poem:

When a programmer is good,
He is very, very good,
But when he is bad,
He is horrid.

Even though the programmers were quite experienced, there was very wide disparity in their abilities. They found the best programmer writes their code about 20 times more quickly than the worst programmer. They debug it 28 times more quickly, the final code runs about 10 times faster, and so on. Follow-up studies report similar disparities, and it has become conventional wisdom that the best computer programmers are about 10 times more productive than the median.

Suppose there is a mediocre and a brilliant computer programmer. Assume that one hour of work by the brilliant programmer is a perfect substitute for ten hours of work by the mediocre programmer. The households are otherwise identical and hold equal shares in the firm.

- (i) Write down a model of this economy, and define a general equilibrium for it.

Answer: **Firm.** Wages w_m and w_b , hours L_m and L_b sale price p , production function f . Profit

$$\pi(p, w_m, w_b) = \max_{L_m, L_b} pf(L_m + 10L_b) - w_m L_m - w_b L_b.$$

Households. Household $h \in \{m, b\}$ chooses consumption c_h and hours l_h to solve

$$\begin{aligned} & \max_{c_h, l_h} u(c_h, l_h) \\ & \text{s.t. } pc_h = w_h l_h + \frac{\Pi}{2}, \end{aligned}$$

where Π is the equilibrium firm profit.

Equilibrium. An equilibrium consists of prices (p^*, w_m^*, w_b^*) and quantities $(c_b^*, c_m^*, l_b^*, l_m^*, L_b^*, L_m^*)$ such that:

- Each decision maker (the households, and the firms) find these quantity choices optimal given prices – see above.
- Consumption clears: $c_m^* + c_b^* = f(L_m^* + 10L_b^*)$.
- The labour markets clear: $L_b^* = l_b^*$ and $L_m^* = l_m^*$.

- (ii) Show that in every equilibrium in which both programmers are hired, the brilliant programmer's wage is ten times higher than the mediocre programmer's wage.

Answer: The firm's FOCs wrt L_b and L_m are, respectively

$$10pf'(L_m + 10L_b) = w_b \quad \text{and} \quad pf'(L_m + 10L_b) = w_m.$$

Dividing the first by the second gives

$$10 = w_b/w_m.$$

(If a worker isn't hired, then we would need a Lagrange multiplier for the constraint of non-negative hours.)

This maths is simply saying: since one hour of brilliant time is a perfect substitute for ten hours of mediocre time, these two options should cost the same. Otherwise, the firm would go for the cheaper option.

- (iii) Show that in every equilibrium, the brilliant programmer is better off than the mediocre programmer.

Answer: The brilliant programmer could make the same choice as the mediocre programmer, and still have money left over to buy more.

- (iv) Depending on the preferences of the households, the brilliant programmer might work longer or shorter hours. Draw the indifference curves in a way that indicates the brilliant programmer working *less* than the mediocre programmer.

- (v) Some people think that the problem is that mediocre programmers are lazy, and they just need some extra incentives to work hard. In the context of your model, would giving the programmers stock options, 100% bonus pay upon project completion and hiring a masseuse and celebrity chef make everyone better off?

Answer: No. By the first welfare theorem, the equilibrium is efficient. Under the feasibility assumptions of the model, there is no allocation that makes everybody better off.

- (vi) The mediocre programmer has another more Machiavellian proposal for increasing productivity. He proposes asking the government to issue a large lump-sum tax on the brilliant programmer, which will force her to work long hours to repay her (government-imposed) debt. The mediocre programmer further proposes the he receive the taxes. Would this proposal work?

Answer: Yes. An allocation in which the mediocre programmer has high consumption supported by the brilliant programmer working very hard is efficient (albeit "unfair"). Thus, by the second welfare theorem, there exist lump-sum taxes to implement this allocation as an equilibrium.

- (vii) * Discuss the problems with proving existence in this economy.

Answer: The firm has a bang-bang solution to hiring workers. If brilliant programmer's wage is not exactly 10 times the mediocre programmer's wage, then the firm will specialise in hiring one of them. Thus, the firm's policy is discontinuous, which is an obstacle to applying Brouwer's fixed point theorem.

5: Micro 1, May 2013

We eat about 300 billion apples every year, but most of these apples can not be eaten directly from the tree. The problem is that apples only ripen in Autumn, and apples consumed at other times must be stored. On the other hand, lettuce may be grown in all seasons, so it is never necessary to store it. Henceforth, assume it is non-storable.

Suppose there are just two seasons (Autumn and Spring) and two foods (lettuces and apples). Farmers are endowed with apples in Autumn, and lettuce in equal quantities in both Autumn and Spring. There is a storage firm (owned by the farmers) that can refrigerate apples until the Spring. The storage technology does not require any labour or other resources to operate. However, as they store more fruit, they become less effective and an increasing fraction of apples go bad.

- (i) Define a general equilibrium in this setting, focusing attention on symmetric equilibria in which all farmers make the same decisions as each other.

Answer: **Farmers:** there are H of them, time $t \in \{1, 2\}$, apple endowment e^A , lettuce endowment e^L , utility function u , apple prices p_1^A, p_2^A and lettuce prices p_1^L, p_2^L , apple consumption a_1, a_2 and lettuce consumption l_1, l_2 , firm profit π .

$$\begin{aligned} & \max_{a_1, a_2, l_1, l_2} u(a_1, a_2, l_1, l_2) \\ \text{s.t.} \quad & p_1^A a_1 + p_2^A a_2 + p_1^L l_1 + p_2^L l_2 = p_1^A e^A + (p_1^L + p_2^L) e^L + \pi/H. \end{aligned}$$

Storage firm: A_1 apples put into storage, $A_2 = f(A_1)$ apples taken out of storage

$$\pi(p_1^A, p_2^A) = \max_{A_1} p_2^A f(A_1) - p_1^A A_1.$$

Market clearing:

$$\begin{aligned} H a_1 + A_1 &= H e^A \\ H a_2 &= A_2 \\ H l_1 &= H e^L \\ H l_2 &= H e^L. \end{aligned}$$

- (ii) Is it possible to normalise apples prices to 1?

Answer: No, it's only possible to normalise one price, e.g. $p_1^A = 1$.

- (iii) Show that if the storage technology is perfect, then apples prices are equal in both seasons.

Answer: Storage firm's first-order conditions:

$$p_2^A f'(A_1) = p_1^A.$$

(This first-order condition holds in any equilibrium in which $a_2 > 0$.) Since $f' = 1$, we conclude that $p_2^A = p_1^A$.

Comment. The intuition behind this mathematics is as follows: If $p_1^A < p_2^A$ then the firm would try to store an infinite amount of apples (so there would be no optimal choice for the firm). If $p_1^A > p_2^A$, then the firm would store nothing.

- (iv) Show if the storage technology involves some spoilage, that apples are more expensive in Spring than Autumn.

Answer: Look at the storage firm's first-order condition (see above). Since there is some wastage, $f'(A_1) < 1$, which means that $p_2^A > p_2^A f'(A_1) = p_1^A$.

- (v) Suppose that the farmers' preferences have a discounted utility representation. (i.e. Time separable preferences that can be written in an additively separable fashion, with per-period utility functions being identical.) Moreover, assume that the farmers have decreasing marginal utility in apple and lettuce consumption. (a) Write the farmers' first-order conditions, (b) show that the farmers consume more apples in Autumn than Spring, and (c) write the farmer's problem using a Bellman equation.

Answer: Discounted utility representation:

$$u(a_1, l_1) + \beta u(a_2, l_2).$$

- (i) Farmers' first-order conditions:

$$\begin{aligned} u_1(a_1, l_1) &= \lambda p_1^A \\ u_2(a_1, l_1) &= \lambda p_1^L \\ \beta u_1(a_2, l_2) &= \lambda p_2^A \\ \beta u_2(a_2, l_2) &= \lambda p_2^L. \end{aligned}$$

- (ii) By market clearing and symmetry, we know that $l_1 = l_2$. Therefore, we have that

$$\lambda = \frac{u_1(a_1, l_1)}{p_1^A} = \frac{\beta u_1(a_2, l_1)}{p_2^A}.$$

Since $p_2^A > p_1^A$ (see the previous question), we deduce that

$$u_1(a_1, l_1) < u_1(a_2, l_1).$$

Since $u_1(\cdot, l_1)$ is decreasing due to decreasing marginal utility, we conclude that $a_2 < a_1$.

- (iii) Let m be money saved for the second period. Bellman equation:

$$\begin{aligned} \max_{a_1, l_1, m} & u(a_1, l_1) + \beta V(m) \\ \text{s.t.} & p_1^A a_1 + p_1^L l_1 + m = p_1^A e^A + p_1^L e^L + \frac{\pi}{H}, \end{aligned}$$

where the second period value function is

$$\begin{aligned} V(m) &= \max_{a_2, l_2} u(a_2, l_2) \\ \text{s.t.} & p_2^A a_2 + p_2^L l_2 = m + p_2^L e^L. \end{aligned}$$

(vi) Now suppose that one farmer is extra productive, and has double the endowments of all of the other farmers. The other farmers have a smaller endowment so that the aggregate endowments are identical. Think about the prices in the following scenarios:

- (a) The original symmetric equilibrium.
- (b) The new equilibrium (with the extra productive farmer).
- (c) A new equilibrium (with the extra productive farmer) in which the productive farmer is taxed so that the equilibrium allocation is the same as in (a).

Do any of these scenarios share the same equilibrium prices?

Answer: Yes, scenarios (a) and (c) by the Second Welfare Theorem.

(vii) Show that the farmers' second-period value function is concave and ** differentiable.

Answer: First, V is concave. Suppose a'_2, l'_2 are optimal choices at m' , and a''_2, l''_2 are optimal choices at m'' . Then for any $t \in [0, 1]$,

$$\begin{aligned}
 & V(tm' + (1-t)m'') \\
 & \geq u(ta'_2 + (1-t)a''_2, tl'_2 + (1-t)l''_2) \\
 & \geq tu(a'_2, l'_2) + (1-t)u(a''_2, l''_2) \\
 & = tV(m') + (1-t)V(m'').
 \end{aligned}$$

Second, by the Benveniste-Scheinkman theorem, V is differentiable at $m > 0$.

6: Micro 1, May 2013

Suppose there are two countries of equal population. However, the big country has twice the amount of land, so that each household located there has twice the land endowment of households in the small country. Each country has an agricultural firm that transforms labour and land into food. Food can be traded on the international market. However, labour and land are more complicated. Each firm is owned equally by the citizens of its own country, and can only grow food on its own country's land. We say that workers migrate if they work for the other country's firm, although we assume that migration is costless.

- (i) Write down a general equilibrium model of the labour, food and land markets. (Hint: treat labour and food as unified international markets, but land as national markets.)

Answer: **Households:** from country $i \in \{0, 1\}$ where 1 is big and 0 is small, food consumption x_i , food price p , labour supplied h_i , wages w , land rental price r_i , land endowment e_i , profit of own country's firm π^i , utility function u , number of households in each country N ,

$$\begin{aligned} \max_{x_i, h_i} u(x_i, h_i) \\ \text{s.t. } px_i = wh_i + r_i e_i + \pi^i / N. \end{aligned}$$

Firms: land rented by firm i is L_i , labour hired H_i , food produced $X_i = f(L_i, H_i)$.

$$\pi^i(p, w, r_i) = \max_{L_i, H_i} pf(L_i, H_i) - wH_i - r_i L_i.$$

Market clearing:

$$\begin{aligned} Ne_0 &= L_0 \\ Ne_1 &= L_1 \\ Nh_0 + Nh_1 &= H_0 + H_1 \\ Nx_0 + Nx_1 &= X_0 + X_1. \end{aligned}$$

Equilibrium. An equilibrium is a vector of quantities $(x_0^*, x_1^*, h_0^*, h_1^*, X_0^*, X_1^*, H_0^*, H_1^*, L_0^*, L_1^*)$ and prices r_0^*, r_1^*, w^*, p^* such that the quantities solve the households' and firms' problems above.

- (ii) Suppose that at some (out-of-equilibrium) prices, the food and labour markets clear, but there is excess demand of the small country's land. What does Walras' law say about the market for the large country's land?

Answer: Walras' law says that if there is excess demand in one market, then there is excess supply in another market. By process of elimination, there must be excess supply of land in the large country at these prices.

- (iii) Show that the small country's firm's profit function is convex in prices.

Answer: The profit function is the upper envelope of linear functions, (one function for each input choice). Therefore it is convex.

- (iv) Show that if wages increase, the small country decreases its demand for labour.

Answer: By the envelope theorem,

$$\frac{\partial \pi^0(p, w, r_0)}{\partial w} = -H_0(p, w, r_0).$$

Since the profit function is convex, both sides of this equation are increasing in w . We conclude that labour demand decreases when wages increase.

- (v) Show that if the production technology has constant returns to scale, and leisure is a normal good, then there is some migration from the small to the big country. (Hint: functions that are homogeneous of degree 1, i.e. satisfy the property that $f(tx, ty) = tf(x, y)$, also have the property that $f_x(2x, 2y) = f_x(x, y)$ for all (x, y) .)

Answer: The firms' labour first-order conditions are:

$$\begin{aligned} pf_H(L_0, H_0) &= w \\ pf_H(L_1, H_1) &= w. \end{aligned}$$

Constant returns to scale implies that f is homogeneous of degree 1. Since $L_1 = 2L_0$, it follows that

$$f_H(L_1, H_1) = f_H(L_0, H_1/2).$$

The ratio of the labour first-order conditions becomes

$$1 = \frac{f_H(L_0, H_0)}{f_H(L_1, H_1)} = \frac{f_H(L_0, H_0)}{f_H(L_0, H_1/2)},$$

which implies that $H_1 = 2H_0$, i.e. the big country's firm hires twice as many worker hours as the small country's firm.

The firms' land first-order conditions are

$$\begin{aligned} pf_L(L_0, H_0) &= r_0 \\ pf_L(L_1, H_1) &= r_1. \end{aligned}$$

Since $(L_1, H_1) = 2(L_0, H_0)$, we deduce that $r_0 = r_1$. This means that the workers in the big country have more non-labour income (land prices are the same but endowments bigger, and profits are bigger in the big country's firm), so they work less as leisure is a normal good. It follows that there is net migration from the small to the big country.

- (vi) Suppose the two countries plan to federalise into a free-trade zone (like the EU). They are worried about social tensions arising from the inequality of the people from the two countries. Devise a lump-sum tax scheme that creates perfect equality.

Answer: The target allocation (of perfect equality) is efficient, so the Second Welfare Theorem implies that lump sum taxes may implement this allocation. Moreover, the theorem describes the transfers needed. Citizens of each country are given a transfer that is equal to the the market value of their equilibrium consumption (i.e. with perfect equality) less the market value of their endowment. This difference is negative for citizens of the big country.

- (vii) * Suppose that households are constrained to work in one country only (of their choice). Discuss how this possibility impedes application of the Brouwer's fixed point theorem to establish existence of equilibria.

Answer: The households no longer have a choice from a convex subset of \mathbb{R}^n , because they have a discrete choice about which country to live in. This isn't necessarily a serious problem, however, since Brouwer's fixed point theorem is typically applied in price space, not consumption space. It might make it difficult to prove continuity of the policy functions though (eg: Berge's theorem of the maximum no longer applies.)

7: Micro 1, December 2013

US comedian Lewis Black has the following to say about solar energy:

If you ask your congressman why, he'll say "Because it's hard. It's really hard. Makes me want to go poopie." You know why we don't have solar energy? It's because the sun goes away each day, and it doesn't tell us where it's going!

Two countries are endowed with some electricity during the day time. However, they are located on opposite sides of the world, so when it is day time in one country, it is night time in the other. Electricity is non-storable, so the only way to consume electricity at night is to import electricity from the other country. A portion of the electricity is lost in transportation; the fraction lost increases as the amount of electricity transported increases.

Apart from this, the countries are identical: there is one household in each country, they share the same preferences and endowments, and the household in each country owns its own electricity exporter. You may assume preferences are additively separable across time, and they value electricity consumption equally during the day and night with decreasing marginal utility.

- (i) Write down a general equilibrium model of this economy for one 24-hour period consisting of one night and day in each country. (Hint: treat electricity in different countries and different times as separate markets.)

Answer: **Households:** country $i \in \{A, B\}$, time $t \in \{1, 2\}$, electricity consumption c_t^i , electricity endowment e_t^i , utility function u , local electricity price p_t^i

$$\begin{aligned} \max_{c_1^i, c_2^i} & u(c_1^i) + u(c_2^i) \\ \text{s.t.} & p_1^i c_1^i + p_2^i c_2^i = p_1^i e_1^i + p_2^i e_2^i + \pi^i. \end{aligned}$$

The question imposes the assumptions that $e_1^B = 0$ and $e_2^A = 0$ and $e_1^A = e_2^B$.

Exporter from country A: x_t^A electricity exported from country A in time t , $y_t^B = f(x_t^A)$ electricity imported into country B in time t ,

$$\pi^A(p_1^A, p_1^B) = \max_{x_1^A} p_1^B f(x_1^A) - p_1^A x_1^A$$

Exporter from country B:

$$\pi^B(p_2^A, p_2^B) = \max_{x_2^B} p_2^A f(x_2^B) - p_2^B x_2^B$$

Market clearing.

$$\begin{aligned} c_1^A + x_1^A &= e_1^A \\ c_1^B &= y_1^B \\ c_2^B + x_2^B &= e_2^B \\ c_2^A &= y_2^A. \end{aligned}$$

Equilibrium. An equilibrium is a vector of quantities $c_1^{*A}, c_2^{*A}, c_1^{*B}, c_2^{*B}, x_1^{*A}, x_2^{*B}, y_1^{*B}, y_2^{*A}$ and prices $p_1^{*A}, p_2^{*A}, p_1^{*B}, p_2^{*B}$ such that the quantities solve the households' and exporters' problems above, and the markets clear.

- (ii) It is possible to eliminate equilibrium variables and conditions using (i) price normalisation and (ii) Walras' law. Provide specific examples of how each of these may be done in the context of your model.

Answer: We may (i) normalise $p_2^B = 1$, and (ii) drop the market clearing constraint

$$c_2^A = y_2^A.$$

- (iii) Suppose that both distributors discover a perfect transportation technology that prevents any electricity from being lost in transportation. In this case, show that both countries have the same sequence of electricity prices.

Answer: If any electricity is exported, then the first-order conditions for the two distributors apply, and they are:

$$\begin{aligned} p_1^B f'(x_1^A) &= p_1^A \\ p_2^A f'(x_2^B) &= p_2^B. \end{aligned}$$

Since no electricity is lost, $f' = 1$, so we conclude that $p_1^A = p_1^B$ and $p_2^A = p_2^B$.

- (iv) Show that if the distributors have a perfect transportation (as above), then the prices are the same. (Hint: look at the households' first-order conditions, and check the market clearing conditions.)

Answer: Since prices are the same in both countries, we write p_1 and p_2 . The households' first-order conditions are

$$\begin{aligned} u'(c_1^A) &= \lambda^A p_1 \\ u'(c_2^A) &= \lambda^A p_2 \\ u'(c_1^B) &= \lambda^B p_1 \\ u'(c_2^B) &= \lambda^B p_2, \end{aligned}$$

which imply

$$\frac{p_1}{p_2} = \frac{u'(c_1^A)}{u'(c_2^A)} = \frac{u'(c_1^B)}{u'(c_2^B)}.$$

This means that if $p_1 > p_2$, then both households consume less electricity in the first period than the second. But this is infeasible, since the aggregate electricity endowment is equal in both periods.

- (v) Consider the proposal of taxing electricity consumption to subsidise electricity distributors to compensate them for the wasted energy lost. Would this proposal make everybody better off?

Answer: No. By the first welfare theorem, every competitive equilibrium is efficient. Therefore, it is not possible to make everybody better off without changing the set of feasible allocations.

- (vi) Again, suppose that there is a perfect transportation technology (see above). Consider the proposal of one country to invade the other, and to impose a new lump-sum tax on the victim country's household. The booty is distributed to the invading country's household. Does this make the invading household better off?

Answer: Yes. Applying the first welfare theorem to the old and new equilibria (before and after the invasion), we know that both equilibria are efficient. Before invasion, both households have equal welfare (since they have the same preferences and budget constraint – see above). After invasion, the invading household has higher utility than the invaded, so it must be better off than before (otherwise, this would be Pareto dominated by the before-invasion allocation).

8: Micro 1, December 2013

Suppose there are two types of people: words people and numbers people. A medicine factory hires workers into two professions: marketing and engineering. Both types of people can do both types of jobs, but words people are better at marketing, and numbers people are better at engineering. Specifically, one hour of a words person's time spent on marketing is equivalent to two hours of a numbers person's time spent on marketing, and vice versa. Both types of people have the same preferences, and are indifferent between both professions – they just take the best wage they can find. Everybody knows what type of person they are trading with.

- (i) Define an equilibrium for this economy.

Comment: The most common mistake was to assume that wages depended on profession rather than skill. (It's possible to prove that it only depends on skill when the worker has no preference about profession.) It would also be ok to have a different wage for every combination of profession and skill.

Another common mistake was to assume that all words people would be assigned to marketing, and all numbers people to engineering. This depends on the firm's production function – perhaps marketing only plays a minor role in the firm, and the firm needs many people working on engineering – even incompetent workers! Incompetent engineers should get paid less than competent ones, so the wages are not based on profession, but rather on skill in this model. (Wages would depend on both if workers disliked one form of work more than another.) One way to avoid this trap is to think about extreme situations. What if the firm only needs one marketing person? Would the firm still want to hire more words people? If you can't think of a reason why not, then you should accommodate it in the model. Also, part (iii) gave the game away – the allocation problem indicates that how to allocate skills to professions is an important trade-off for the problem. Therefore, it's worth reading the whole question to understand the spirit of it, to make sure you aren't missing something important.

Answer: Workers. Worker type $t \in \{N, W\}$, number of type t workers n_t , medicine consumed m_t , medicine price p , hours of labour supplied h_t , wage w_t , firm profit π , utility $u(h_t, m_t)$.

$$\begin{aligned} & \max_{h_t, m_t} u(h_t, m_t) \\ & \text{s.t. } pm_t = w_t h_t + \frac{\pi}{n_N + n_W}. \end{aligned}$$

Factory. Profession $s \in \{E, M\}$, type t worker hours allocated to profession s is written H_{ts} , labour inputs $(H_N, H_W) = (H_{NE} + H_{NM}, H_{WE} + H_{WM})$, medicine produced $M = f(2H_{NE} + H_{WE}, H_{NM} + 2H_{WM})$, profit π given by

$$\begin{aligned} & \max_{H_{NE}, H_{WE}, H_{NM}, H_{WM}} pf(2H_{NE} + H_{WE}, H_{NM} + 2H_{WM}) \\ & \quad - w_N(H_{NE} + H_{NM}) - w_W(H_{WE} + H_{WM}) \end{aligned}$$

Equilibrium. $(p^*, w_W^*, w_N^*, h_W^*, h_N^*, m_W^*, m_N^*, H_{NE}^*, H_{WE}^*, H_{NM}^*, H_{WM}^*, M^*)$ form an equilibrium if the household's and firm's respective choices are optimal as defined above, and the following market clearing conditions are satisfied:

$$\begin{aligned} N_W m_W^* + N_N m_N^* &= M^* \\ N_W h_W^* &= H_W^* \\ N_N h_N^* &= H_N^*. \end{aligned}$$

- (ii) Suppose there is excess demand for both types of labour, i.e. at market prices, the firm demands more labour than the workers are willing to supply. Does this mean that there is also excess demand for medicine?

Answer: No. Walras' law implies that there is excess supply of medicine.

- (iii) The factory has to make two types of choices: how many workers of each type to hire, and how to allocate them to professions.
- Define the firm's output function as the maximum amount of medicine the firm can produce with given labour inputs.
 - Write down a Bellman equation for the factory relating the firm's cost function to the firm's output function.
 - Show that the firm's cost function is concave with respect to wages.
 - Show that if the market wage of numbers people increases, then the firm finds it optimal to meet its production target by hiring fewer numbers people and more words people.

Answer:

- (a)

$$\begin{aligned} F(H_N, H_W) &= \max_{H_{NE}, H_{WE}, H_{NM}, H_{WM}} f(2H_{NE} + H_{WE}, H_{NM} + 2H_{WM}) \\ \text{s.t. } &H_{NE} + H_{NM} = H_N \text{ and } H_{WE} + H_{WM} = H_W. \end{aligned}$$

- (b)

$$\begin{aligned} c(M; w_N, w_W) &= \min_{H_N, H_W} w_N H_N + w_W H_W \\ \text{s.t. } &F(H_N, H_W) = M. \end{aligned}$$

- (c) Holding the output target M fixed, the firm's cost function is the lower envelope of a set of linear functions (one function for each feasible pair (H_N, H_W) that can be used to meet the target). The lower envelope of linear functions (which are concave) is concave.
- (d) By the envelope theorem,

$$\frac{\partial c(M; w_N, w_W)}{\partial w_N} = H_N(M; w_N, w_W).$$

Since the cost function is concave, the left side is decreasing in numbers wages w_N . It follows that the right side, the number of numbers people hired $H_N(M; w_N, w_W)$ to meet output target M , is a decreasing function of numbers wages w_N . Therefore, more words people must be hired to meet the target.

- (iv) Suppose the Words Union has an agreement which guarantees a maximum number of hours for words people only, and that this makes the words people better off. The Numbers Union proposes offering the Words Union a deal: it would tax numbers workers a little bit, and give those taxes to words workers. In return, the Words Union would abandon its maximum hours policy. Is it possible that both unions would agree to this deal?

Answer: Yes. There are three relevant allocations to consider, (i) the competitive equilibrium, (ii) the Words Union allocation, and (iii) the lump-sum tax allocation. By the first welfare theorem, allocation (i) is efficient. Since words people are better off in (ii) than (i), the numbers people must be worse off in (ii) than (i). On the other hand, allocation (ii) need not be efficient. It might be Pareto dominated by another allocation, and hence dominated by an efficient allocation, which we might call (iii). By the second welfare theorem, allocation (iii) can be implemented by lump-sum taxes. Conclusion: if the Numbers Union deal is inefficient, then a deal involving lump-sum taxes to cancel the agreement is Pareto improving, and would be accepted by both unions.

- (v) * Prove that the cost function is differentiable with respect to wages.

Answer: We already established that the cost function is concave with respect to wages. Hold the output target M^* fixed, and pick any pair of wages, (w_N^*, w_W^*) . For these wages and output target, there is an optimal hours choice, (H_N^*, H_W^*) , and the “lazy” cost function

$$\bar{c}(M^*; w_N, w_W) = H_N^* w_N + H_W^* w_W$$

is a differentiable upper support function for the cost function at (w_N^*, w_W^*) . Therefore, by the Benveniste-Scheinkman theorem, the cost function is differentiable at (w_N^*, w_W^*) . But the choice of these wages was arbitrary, so the cost function is differentiable everywhere.

9: Micro 1, December 2013

A child care centre provides any number of hours of care to several households using two types of labour: babysitters and cleaners. Both types of labour are necessary for production – if either is zero, then no care can be provided. Households can simultaneously supply labour of both types. Households are also endowed with divisible houses, which they can exchange.

Comment: The main difficulties students have with this question are the welfare parts (iv) and (v). The thrust of the question is: do the welfare theorems apply when specialisation is required? You have to know the proofs of the welfare theorems to answer these questions well. The proof of the first welfare theorem does not really require a convex budget constraint (see the sample solution for details), but the second welfare theorem uses it.

- (i) Define the concept of a symmetric equilibrium for this economy, in which each household makes the same choice.

Answer: Households. N number of households, b labour on babysitting, c labour on cleaning, w_b wage for babysitting, w_c wage for cleaning, p care price, x child-care services demanded, h housing demand, e housing endowment, q house price, $u(h, x, b, c)$ utility, π firm profits,

$$\begin{aligned} & \max_{h,x,b,c} u(h, x, b, c) \\ \text{s.t. } & qh + px = qe + w_b b + w_c c + \frac{\pi}{N}. \end{aligned}$$

Firm. B baby sitters hired, C cleaners hired, $X = f(B, C)$ care output,

$$\pi(p, w_b, w_c) = \max_{B,C} pf(B, C) - w_b B - w_c C.$$

Equilibrium. $(q^*, p^*, w_b^*, w_c^*, h^*, x^*, b^*, c^*, X^*, B^*, C^*)$ form an equilibrium if the households' and firm's respective choices are optimal, as defined above, and the following market clearing conditions are satisfied:

$$\begin{aligned} Nh^* &= Ne^* \\ Nx^* &= X^* \\ Nb^* &= B^* \\ Nc^* &= C^*. \end{aligned}$$

- (ii) Suppose at all equilibrium allocations, the households have a higher marginal utility loss of cleaning than babysitting. Show that in every equilibrium, the cleaning wage is higher than the babysitting wage.

Answer: Let λ be the Lagrange multiplier for the budget constraint. The household's first-order conditions with respect to cleaning and babysitting are

$$\begin{aligned} -\frac{\partial u(h, x, b, c)}{\partial b} &= \lambda w_b \\ -\frac{\partial u(h, x, b, c)}{\partial c} &= \lambda w_c. \end{aligned}$$

On the left side, the first line is lower than the second by the assumption. And the right side, it follows that $w_b < w_c$.

- (iii) Suppose that the firm's production function is not concave. Does this imply that the profit function is not convex in prices?

Answer: No, it is still convex! The profit function is linear in prices, because it is the upper envelope of linear functions. Specifically for each input vector (B, C) , the function

$$g(p, w_b, w_c; B, C) = pf(B, C) - w_b B - w_c C$$

is linear in (p, w_b, w_c) , and the profit function is the upper envelope of all g functions.

- (iv) Suppose that workers must specialise in at most one profession, babysitting or cleaning. (This isn't a government restriction, just a difficulty of working in these professions.) Are all equilibria efficient? Specifically, is it the case that every equilibrium in this environment is Pareto undominated by every feasible allocation in this environment?

Answer: Yes. The proof of the first welfare theorem is based on the idea that if an allocation Pareto dominates an equilibrium allocation, then that allocation is more valuable at the market prices of the equilibrium allocation, and is therefore infeasible. This proof only applies directly to pure-exchange economies, but can be extended to production economies using the idea of home production. Adding a specialisation constraint would not be a problem for the proof. In particular, it would not affect the key step that at least one household must be unable to afford its consumption in the supposedly Pareto dominating allocation.

- (v) * As in the previous part, suppose that workers must specialise in at most one profession, babysitting or cleaning. Can every efficient allocation in this environment be implemented using lump-sum taxes?

Answer: No. First, the proof in class does not apply. It relies on the existence theorem, which is inapplicable since the excess demand function is not continuous: a small change in relative wages could make the household make a discontinuous switch in specialisation. Second, existence is essential – if there is no equilibrium when the endowment equals the efficient allocation, then there will be no way to implement that allocation in a competitive equilibrium with lump-sum taxes.

10: Micro 1, May 2014

Suppose there are two rural districts that share an identical agricultural technology for transforming water into food. In the first year, households in both districts are endowed with the same amount of water, which they sell to farms. In the second year, one district suffers a perfectly predictable drought and has no water endowment. Households only directly consume food, and only hold shares in local farms. There are no import/export or migration costs, but food and water are non-storable.

- (i) Write down a competitive general equilibrium model of the economy. You may assume households' preferences can be represented by an additively separable utility function.

Comment: Make sure you get your markets right! (There are 4 markets: food and water in periods 1 and 2). It's not a problem if you have extra markets (eg: food in district A in period 1) as long as the logic of your model implies the prices are equal across your artificial markets, and that it is feasible within your economy for food to be reallocated between districts. (Eg: each firm can sell their output to both districts, which would imply the prices are equal – otherwise, firms would specialise in one district).

Make sure you get your choice variables (under the max) right! Many students write that the water endowments were choice variables. I imagine the source of confusion is that students expect the households to have to choose something about water – but get confused when the households didn't consume their water. The most straightforward answer is to assume the households have NO choice – they sell all of their water. Another option is to separately account for the endowment of water and consumption of water, and since the household derives no utility from its consumption, it will sell all of it.

Usually, every cost should have a corresponding benefit (and vice versa). In this question, we have an exception: there is no cost to households of giving up their water endowments. But this makes sense (it was in the question). It's good to double check: have all my costs got benefits?

Answer: Households. Districts $d \in \{A, B\}$ where B suffers the drought, year $t \in \{1, 2\}$, number of households N_d , food consumption c_{dt} , water endowment w_{dt} (the drought makes $w_{B2} = 0$), food price p_t , water price s_t , discount rate β , per-period utility $u(c_{dt})$, farm profit π_d :

$$\begin{aligned} & \max_{c_{d1}, c_{d2}} u(c_{d1}) + \beta u(c_{d2}) \\ \text{s.t. } & p_1 c_{d1} + p_2 c_{d2} = s_1 w_{d1} + s_2 w_{d2} + \frac{\pi_d}{N_d}. \end{aligned}$$

Farms. Water demand of farm located in district d is W_{dt} , production function $f(W_{dt})$

$$\pi(p_1, p_2, s_1, s_2) = \max_{W_{d1}, W_{d2}} p_1 f(W_{d1}) + p_2 f(W_{d2}) - s_1 W_{d1} - s_2 W_{d2}.$$

Equilibrium. $(p_t^*, s_t^*, c_{dt}^*, w_{dt}^*, W_{dt}^*)$ form an equilibrium if the households' and firm's respective choices are optimal, as defined above, and the following market clearing conditions are satisfied:

$$\begin{aligned} N_A^* c_{A1}^* + N_B^* c_{B1}^* &= f(W_{A1}^*) + f(W_{B1}^*) \\ N_A^* c_{A2}^* + N_B^* c_{B2}^* &= f(W_{A2}^*) + f(W_{B2}^*) \\ N_A^* w_{A1}^* + N_B^* w_{B1}^* &= W_{A1}^* + W_{B1}^* \\ N_A^* w_{A2}^* + N_B^* w_{B2}^* &= W_{A2}^* + W_{B2}^* \end{aligned}$$

- (ii) Suppose that some protesters succeed in lowering the price of water in the second period, which leads to excess demand of water in the second period. According to Walras' law, what other consequences would this non-equilibrium behaviour have?

Comment: Students often incorrectly apply Walras' law by identifying a specific market with excess supply.

Answer: If there's excess demand in one market, there must be excess supply in another market. However, Walras' law does not say which market this might occur in.

- (iii) Show that each household has a decreasing marginal value of saving for the second year, provided that the household has a decreasing marginal utility of consumption. (Hint: this involves formulating the value of savings.)

Answer: The value of savings m in the second year is

$$V_{d2}(m; p_2, s_2) = u\left(\frac{m + s_2 w_{d2}}{p_2}\right).$$

V_{d2} is a concave function in m , because it is the composition of a concave function u with a linear function. It's derivative, the marginal value of savings, is therefore a decreasing function.

- (iv) Show that each household consumes less during the drought.

Answer: The first-order conditions for a household in district d can be simplified to

$$\lambda_d = \frac{u'(c_{d1})}{p_1} = \beta \frac{u'(c_{d2})}{p_2},$$

where λ_d is the Lagrange multiplier for the budget constraint. Since output in the second year, $f(W_2)$ is less than output in the first year, $f(W_1)$, at least one household consumes less in the second year. So that household, in district d , has (by decreasing marginal utility)

$$\frac{u'(c_{d1})}{u'(c_{d2})} < 1.$$

By the first-order condition, the left side of this inequality (the marginal rate of substitution of consumption between the two periods) is the same for all households in equilibrium:

$$\beta \frac{p_1}{p_2} = \frac{u'(c_{d1})}{u'(c_{d2})}.$$

Therefore, all households satisfy the inequality, and hence consume less in the second period.

- (v) The government would like to compensate the drought-stricken district. Either devise a lump-sum tax policy that would implement smooth (constant) consumption over time for all households, or prove that this task is impossible.

Answer: It is impossible. Any allocation that involves constant consumption over time for all households is *inefficient*, since output is higher in the first period than the second. By the first welfare theorem, any competitive equilibrium is *efficient* (regardless of how endowments are reallocated). Therefore, regardless of the lump-sum taxes chosen, the competitive equilibrium would not involve constant consumption.

- (vi) * Write down a function that has the following property: a price vector is a fixed point of that function if and only if there exists an equilibrium with that price vector. Your function should never lead to negative prices. (You may make use of the excess demand function without defining it explicitly.)

Answer: Let $P = (p_1, p_2, s_1, s_2)$ denote a price vector and let $z(P)$ denote the excess demand function. Then the function

$$\phi(P) = \begin{bmatrix} \max \{P_1, P_1 + z_1(P)\} \\ \dots \\ \max \{P_4, P_4 + z_4(P)\} \end{bmatrix}$$

has the required property. If P has an equilibrium allocation, then $z(P) = 0$ and hence $\phi(P) = P$. Conversely, if P does not have an equilibrium allocation, then by Walras' law, there is excess demand in one market (and excess supply in another market), so $\phi(P) \neq P$.

11: Micro 1, May 2014

Individuals are endowed with one unit of human capital and time. In the first year, individuals divide their time between accumulating human capital (through self-study), labour, and leisure. In the second year, the individuals divide their time between labour and leisure only. A firm produces a consumption good in each year using labour. The contribution of each hour of work to production is proportional to the worker's human capital.

- (i) Write down a perfectly competitive model for this market. You may assume the households have additively separable utility, with stationary flow utility. (Hint: the human capital production function should have decreasing marginal product.)

Comment: A common mistake is to have labour and leisure as separate goods. You can split them if you like, but then you should have a time budget constraint.

Answer: Households. Time $t \in \{1, 2\}$, number of households N , consumption c_t , human capital endowment $k = 1$, human capital investment i , human capital production function $g(i)$, labour supply l_t , consumption price p_t , wages w_t , flow utility $u(\cdot, \cdot)$, discount rate β , equilibrium firm profit π . Households solve

$$\begin{aligned} \max_{c_1, c_2, i, l_1, l_2} \quad & u(c_1, l_1 + i) + \beta u(c_2, l_2) \\ \text{s.t.} \quad & p_1 c_1 + p_2 c_2 = w_1 k l_1 + w_2 (k + g(i)) l_2 + \frac{\pi}{N}. \end{aligned}$$

Firm. Labour demand L_t , production function $f(L_t)$, profit maximisation problem:

$$\pi(p_1, p_2, w_1, w_2) = \max_{L_1, L_2} p_1 f(L_1) + p_2 f(L_2) - w_1 L_1 - w_2 L_2.$$

Equilibrium. $(p_1^*, p_2^*, w_1^*, w_2^*, k^*, i^*, c_1^*, c_2^*, l_1^*, l_2^*, L_1^*, L_2^*)$ forms an equilibrium if the choices solve the household's and firm's problem, and markets clear, i.e.

$$\begin{aligned} N c_1^* &= f(L_1^*) \\ N c_2^* &= f(L_2^*) \\ N k l_1^* &= L_1^* \\ N (k + g(i^*)) l_2^* &= L_2^*. \end{aligned}$$

- (ii) Is it possible for the price of consumption in the first period to be 1?

Answer: Yes. If $P^* = (p_1^*, p_2^*, w_1^*, w_2^*)$ is an equilibrium price vector, then so is P^*/p_1^* .

- (iii) Write down a value function for the start of the second year. (Hint: the state variable includes human capital, savings, and the prices in the second year.)

Answer.

$$\begin{aligned} V(k_2, m_2; p_2, w_2) &= \max_{c_2, l_2} u(c_2, l_2) \\ \text{s.t.} \quad & p_2 c_2 = w_2 k_2 l_2 + m_2. \end{aligned}$$

(iv) Derive the marginal value of (a) human capital and (b) savings.

Answer. (a) Substituting the budget constraint into the objective gives

$$V(k_2, m_2; p_2, w_2) = u((w_2 k_2 l_2(k_2, m_2; p_2, w_2) + m_2)/p_2, l_2(k_2, m_2; p_2, w_2)).$$

By the envelope theorem,

$$\begin{aligned} \frac{\partial V(k_2, m_2; p_2, w_2)}{\partial k_2} &= \left[\frac{\partial}{\partial k_2} u((w_2 k_2 l_2 + m_2)/p_2, l_2) \right]_{l_2=l_2(k_2, m_2; p_2, w_2)} \\ &= \left[u_c((w_2 k_2 l_2 + m_2)/p_2, l_2) \frac{w_2 l_2}{p_2} \right]_{l_2=l_2(k_2, m_2; p_2, w_2)} \\ &= u_c(c_2(k_2, m_2; p_2, w_2), l_2(k_2, m_2; p_2, w_2)) \frac{w_2 l_2(k_2, m_2; p_2, w_2)}{p_2}. \end{aligned}$$

(b) A similar procedure gives

$$\frac{\partial V(k_2, m_2; p_2, w_2)}{\partial m} = u_c(c_2(k_2, m_2; p_2, w_2), l_2(k_2, m_2; p_2, w_2)) \frac{1}{p_2}.$$

(v) The government thinks that it's wasteful for everybody to become educated. It proposes a tax on labour earnings in the second year to encourage more labour to be supplied in the first year. Could such a policy be Pareto-improving?

Answer. No. By the first-welfare theorem, the equilibrium (without any taxes) is efficient, so no Pareto-improving allocations are feasible.

(vi) * Informally discuss whether there are any asymmetric equilibria (e.g. in which some people choose to become well-educated, but others do not.)

Answer. Typically, the household's optimisation problem has a unique solution (because the objective is concave and the feasible choices lie in a convex set). When this is the case, all households have the same problem, and hence the same solution. In this model (as formulated in these sample solutions), the human capital multiplies hours worked in a non-convex way, so households might be indifferent between several choices. This could lead to multiple equilibria.

12: Micro 1, December 2014

A factory produces appliances using labour and waste disposal services. Households supply labour and waste disposal. Households are endowed with small or large gardens, where they can dispose of waste. Assume that households do not suffer from storing waste in their gardens, and that gardens are not traded (or at least, not directly).

- (i) Write down a competitive model of the labour, appliance, and waste disposal markets.

Comment: A common mistake is to (implicitly) assume that households with big and small gardens made the same choices. You can't just write c for consumption, because people with bigger gardens will consume more. There are several alternatives. You could write c_h for household h 's consumption, or you could write c_B for the big garden's consumption. (Or you could write the garden endowment as a parameter to the optimisation problem, and write down a policy function...) The most important thing is that the market clearing conditions (for all markets) accommodate people with different garden sizes making different choices.

It is also possible to formulate the consumer's problem so that the household can consume gardens in addition to selling them, e.g. by playing football. But they would not derive any utility from football, as per the question.

Answer: Consumer's problem. Notation: $h \in \{1, \dots, N\}$ household address, a_h appliance choice, p price of appliances, l_h labour, w wages, g_h garden capacity, r price of disposal services, $u(a_h, l_h)$ utility, π firm profit (see below)

$$\max_{a_h, l_h} u(a_h, l_h) \quad (12)$$

$$\text{s.t. } pa_h = wl_h + rg_h + \pi/N. \quad (13)$$

Firm's problem. Notation: L labour demand, T waste supply, $A = f(L, T)$ appliance supply.

$$\pi(p, w, r) = \max_{L, T} pf(L, T) - wL - rT. \quad (14)$$

Market clearing conditions.

$$\sum_h a_h = A \quad (15)$$

$$\sum_h l_h = L \quad (16)$$

$$\sum_h g_h = T. \quad (17)$$

Equilibrium. A price vector (p^*, w^*, r^*) and an allocation

$$(\{a_h^*\}, \{l_h^*\}, A^*, L^*, T^*)$$

forms an equilibrium if the allocation satisfies the market clearing conditions, and the households' and firm's respective allocations solve their respective problems, given the price vector.

- (ii) Show that in every equilibrium, all households' gardens are filled to capacity with waste.

Answer: If the price of waste disposal is greater than zero (i.e. $r > 0$), then there is a benefit, but no cost of filling the garden to capacity.

Alternative answer: If you formulate the household problem with a (useless) consumption choice of garden usage s_h , then an interior solution would satisfy the first-order condition for s_h ,

$$0 = \lambda r,$$

where λ is the Lagrange multiplier on the budget constraint. Since $\lambda > 0$ and $r > 0$ in every equilibrium, this is a contradiction. So the assumption that s_h is an interior solution is false.

- (iii) Show that if leisure is a normal good, then households with bigger gardens work less.

Answer: Households with bigger gardens have more wealth, and therefore consume more leisure (since leisure is a normal good). Which is another way of saying that they work less.

- (iv) Show that if the price of waste disposal increases, then firms will generate less waste.

Answer: First, notice that π is the upper envelope of a set of straight lines, one for each choice (L, T) . Therefore, π is convex. By the envelope theorem

$$\frac{\partial}{\partial r} \pi(p, w, r) = -T(p, w, r), \quad (18)$$

where $T(p, w, r)$ is the demand for waste disposal when prices are (p, w, r) . Since π is convex, the left side is an increasing function in r . Therefore, the right side is also increasing in r , hence $T(p, w, r)$ is decreasing in r .

- (v) Suppose the government wants to decrease the amount of waste stored in gardens. Is there a lump-sum tax scheme that would work?

Answer: No, by part (ii), no matter what the endowments are, all households will fill their gardens to capacity with waste. Therefore, there is no lump-sum tax regime that would work.

- (vi) * Under what conditions would the households have a unique optimal labour, appliance and waste storage choice?

Answer: If all prices are non-zero, and the household's utility function is strictly quasi-concave (or strictly concave), then the household would have only one optimal choice.

- (vii) * Prove that if all prices are greater than zero, and that households can work at most 24 hours per day, then the budget set (i.e. the set of affordable feasible choices) is compact.

Answer: We require $l \in [0, 24]$, so let $F = \mathbb{R}_+ \times [0, 24]$ be the set of feasible choices for the household (before considering the budget constraint).

Let $U_h(a, l) = wl + \pi/N + rg_h - pa$ be the amount of money that is unspent when household h chooses (a, l) . This function is continuous. The set of affordable allocations is $A = (U_h)^{-1}(\mathbb{R}_+)$. Since \mathbb{R}_+ is closed and U_h is continuous, A is closed. The budget set $B = A \cap F$ is the intersection of two closed sets, and is therefore closed.

For any $(a, l) \in B$, we know $l \leq 24$, so $a \leq \frac{1}{p}(24w + \pi/N + rg_h)$. Therefore B is bounded, i.e. contained in some ball.

Since B is closed and bounded, the Bolzano-Weierstrass theorem implies that it is compact.

13: Micro 1, December 2014

As the earth's population grows, an important question is how future inhabitants will be able to feed themselves, and whether this will lead to inter-generational inequality. Suppose there are two generations (X and Y) of equal size. Generation X lives for two time periods, but Generation Y only lives in the second time period. This means that the population is higher in the second period.

Farms produce food using land and labour. Only Generation X is endowed with land, which it can supply to the market. Generation X households hold all of the shares in the farms. Both generations can supply labour and consume food. Households do not benefit from occupying land (but can gain wealth from renting out the land). Generation X has stationary time-separable preferences, and its per-period utility function is the same as Generation Y's.

- (i) Write down a competitive general equilibrium model of this economy.

Comment: Firms are active in two time periods $t \in \{1, 2\}$. A common mistake is to write something like

$$\pi(p_t, w_t) = \max_{x_t} p_t f_t(x_t) - w_t \cdot x_t. \quad (19)$$

This is ambiguous, and both possible interpretations are wrong! One interpretation is that $\pi(p_t, w_t)$ is shorthand for $\pi(p_1, p_2, w_1, w_2)$. (A less ambiguous shorthand is $\pi(\{p_t, w_t\}_{t \in \{1, 2\}})$ or just $\pi(p, w)$.) This interpretation makes no sense, because the objective does not explain how profits are combined from both periods. One way to fix this problem is to instead write

$$\pi(p, w) = \max_x \sum_{t \in \{1, 2\}} [p_t f_t(x_t) - w_t \cdot x_t]. \quad (20)$$

Another interpretation is that there are two firms, one operating in each period. But if this is the case, they should have different profit functions, and in the households' budget constraints, you should be including the dividends of both firms. For example, you might write that the profit function of the firm operating in period t is

$$\pi^t(p_t, w_t) = \max_{x_t} p_t f_t(x_t) - w_t \cdot x_t. \quad (21)$$

Answer: Generation X's problem. Notation: c_t^X food consumption in period $t \in \{1, 2\}$, p_t food price, w_t wage, h_t^X labour supply, r_t land rent, l^X land endowment, $u(c, h)$ per-period utility function, β discount rate, π farm profit (see below), N^X Generation X population.

$$\max_{c_t^X, h_t^X} u(c_1^X, h_1^X) + \beta u(c_2^X, h_2^X) \quad (22)$$

$$\text{s.t. } p_1 c_1^X + p_2 c_2^X = w_1 h_1^X + w_2 h_2^X + (r_1 + r_2) l^X + \pi / N^X \quad (23)$$

Generation Y's problem. Notation: c^Y food consumption, h^Y labour supply, N^Y Generation Y population.

$$\max_{c^Y, h^Y} u(c^Y, h^Y) \quad (24)$$

$$\text{s.t. } p_2 c^Y = w_2 h^Y. \quad (25)$$

Farm's problem. Notation: L_t land demand, H_t labour demand, $C_t = f(L_t, H_t)$ food output in period t .

$$\pi(p_1, p_2, w_1, w_2, r_1, r_2) \quad (26)$$

$$= \max_{L_1, L_2, H_1, H_2} p_1 f(L_1, H_1) + p_2 f(L_2, H_2) - w_1 H_1 - w_2 H_2 - r_1 L_1 - r_2 L_2. \quad (27)$$

Market clearing conditions.

$$C_1 = N^X c_1^X \quad (28)$$

$$C_2 = N^X c_2^X + N^Y c^Y \quad (29)$$

$$L_1 = N^X l^X \quad (30)$$

$$L_2 = N^X l^X \quad (31)$$

$$H_1 = N^X h_1^X \quad (32)$$

$$H_2 = N^X h_2^X + N^Y h^Y. \quad (33)$$

Equilibrium. A price vector $(p_1, p_2, w_1, w_2, r_1, r_2)$ and an allocation

$$(\{(c_t^X, h_t^X)\}_t, c^Y, h^Y, \{(C_t, L_t, H_t)\}_t)$$

is an equilibrium if the households' and firms' allocations are optimal choices given the prices, and the markets clear.

- (ii) Suppose that if the prices in all markets (labour, land, and food) do not increase over time, that there is excess demand of labour, land, and food in the second period. Does this imply that there is excess supply in all markets in the first period?

Answer: No. From Walras' law, we know that at least one market in the first period has excess supply, but it may not be all of them.

- (iii) For this part, focus attention on equilibria in which food output is higher in the second period. Show that in every such equilibrium, real wages (i.e. wages divided by food prices) are lower in the second period.

Answer: In every equilibrium, $L_2 = L_1$ (from the market clearing conditions). Since food output is higher in the second period, this implies $H_2 > H_1$. From the firm's first-order conditions, we can deduce

$$f_H(L_1, H_1) = \frac{w_1}{p_1} \quad (34)$$

$$f_H(L_2, H_2) = \frac{w_2}{p_2}. \quad (35)$$

If f has decreasing marginal productivity, then $H_2 > H_1$ implies

$$f_H(L_1, H_1) > f_H(L_2, H_2). \quad (36)$$

We conclude then that real wages are higher in the first period, i.e.

$$\frac{w_1}{p_1} > \frac{w_2}{p_2}. \quad (37)$$

- (iv) Write down Generation X's value of holding money in the second period. (Hint: this should be a function of money and second period food prices and wages.)

Answer: Generation X's indirect utility function is

$$v(m; p_2, w_2) = \max_{c_2^X, h_2^X} u(c_2^X, h_2^X) \quad (38)$$

$$\text{s.t. } p_2 c_2^X = m + w_2 h_2^X. \quad (39)$$

- (v) Reformulate Generation X's problem by using the value function from (iv) twice, i.e. the household should choose how to allocate money between the two periods. How the money is spent in each period should be buried inside the value function.

Answer: A reformulation of the Generation X problem:

$$\max_{m_1, m_2} v(m_1; p_1, w_1) + \beta v(m_2; p_2, w_2) \quad (40)$$

$$\text{s.t. } m_1 + m_2 = \pi/N^X + (r_1 + r_2)l^X. \quad (41)$$

- (vi) Generation Y protestors would like to eat more and work less, so they propose confiscating land from Generation X at the start of period 2, and giving it to Generation Y. Can such a policy make Generation Y better off? Would the proposal lead Generation Y to eat more and work less?

Answer: Confiscating land is equivalent to lump-sum taxation of the value of that land (at market prices). By the second welfare theorem, any efficient allocation can be implemented by doing this, and some efficient allocations would make Generation Y better off.

However, it's not clear if there is any efficient allocation in which Generation Y both works less and consumes more. (That depends on preferences.)

- (vii) * The proof of existence of equilibrium relies on applying Brouwer's fixed point theorem, which requires a set to be convex (among other things). Economically speaking, which set is convex? Is this assumption usually met?

Answer: Brouwer's fixed point theorem is about a function $f : X \rightarrow X$, and it requires the set X to be convex. Economically speaking, X is the set of possible prices. The requirement that X be convex is very easy to satisfy. In the existence proof, we normalise prices to sum to 1, so the set of possible prices is a straight line (or hyperplane), which is convex.

- (viii) * Holding prices fixed, consider a sequence of optimal labour supply and consumption choices, where the expenditure decreases to 1. Does this sequence have a convergent subsequence (using the Euclidean metric)?

Answer: Let e_n denote the expenditure for the n^{th} choice. Since e_n is decreasing, all choices are contained in the budget set corresponding to expenditure e_1 . Since this budget set is compact, every sequence inside of it has a convergent subsequence.

14: Micro 1, May 2015

Suppose there are two occupations, nursing and cleaning, and that individuals must select only one occupation to work in each year. Cleaning is easy to learn, but nurses with one year of experience become more productive. There are two years in the economy. Hospitals hire nurses and cleaners to provide medical services, and share their profits equally among the population. Individuals consume medical services.

- (i) Write down a competitive model of the nursing and cleaning markets across the two years. (Hint: there are no symmetric equilibria, so you will need to accommodate identical households taking different decisions.)

Comment: This question is a little tricky to formulate well:

- One common mistake is to consider the experience a discrete choice, rather than depending on how hard the nurses work. This is partly my fault – it isn't until part (iv) that this becomes clear.
- The most common mistake is to write down the worker's utility functions conditional on occupation choice, but without studying the worker's decision about which occupation to choose. Despite this, students typically answer part (v) well (which was about workers being indifferent between nursing and cleaning)

Answer: Individuals. There are two fields, $o \in \{C, N\}$, cleaning and nursing. Individual $i \in I$ chooses how many hours to work in cleaning (h_{iC}^i) at wage w_{tC} and nursing (h_{iN}^i) at wage w_{tN} , consumption of medical m_t^i services at prices p_t . The experience-adjusted productivity of nursing in the second period is $x(h_{1N}^i)$, where $x(0) = 1$. The individual has a discount factor β , and utility $u(m_t^i, 1 - h_{iC}^i - h_{iN}^i)$ in each period. Hospital profits (defined below) are Π . Individual i 's problem is:

$$\begin{aligned} & \max_{\{m_t^i\}_t, \{h_{to}^i\}} \sum_{t=1}^2 \beta^t u(m_t^i, 1 - h_{iC}^i - h_{iN}^i) \\ \text{s.t. } & p_1 m_1^i + p_2 m_2^i = w_{1C} h_{1C}^i + w_{1N} h_{1N}^i + w_{2C} h_{2C}^i + w_{2N} x(h_{1N}^i) h_{2N}^i + \frac{\Pi}{|I|}, \\ & \text{and either } h_{iN}^i = 0 \text{ or } h_{iC}^i = 0. \end{aligned}$$

The hospital. The hospital hires H_{tC} cleaner hours and H_{tN} productivity-adjusted nursing hours in time t , and produces $f(H_{tC}, H_{tN})$ units of medical services. Their profits are

$$\Pi(p_t, w_{1C}, w_{2C}, w_{1N}, w_{2N}) = \max_{H_{to}} \sum_t p_t f(H_{tC}, H_{tN}) - \sum_{t,o} w_{to} H_{to}. \quad (42)$$

Equilibrium. An allocation of resources ($\{m_t^{i*}, h_{iN}^{i*}, h_{iC}^{i*}\}, \{H_{to}^*\}$) and prices ($\{p_t^*\}, \{w_{to}^*\}$) constitute an equilibrium if each household and hospital finds this allocation opti-

mal (see above), and the six markets clear, i.e.

$$\sum_i m_1^{i*} = f(H_{1C}^*, H_{1N}^*), \quad (43)$$

$$\sum_i m_2^{i*} = f(H_{2C}^*, H_{2N}^*), \text{ and} \quad (44)$$

$$\sum_i h_{to}^{i*} = H_{to}^* \text{ for } to \in \{1C, 1N, 2C, 2N\}. \quad (45)$$

- (ii) Write down a formula for the value of savings and nursing experience in the second year.

Answer: Let s be savings, and x be nursing experience like before. Individual i 's value function is

$$V_i(s, x) = \max_{m_2^i, h_{2C}^i, h_{2N}^i} u(m_2^i, 1 - h_{2C}^i - h_{2N}^i) \quad (46)$$

$$\text{s.t. } p_2 m_2^i = w_{2C} h_{2C}^i + w_{2N} x h_{2N}^i + s, \quad (47)$$

$$\text{and either } h_{2N}^i = 0 \text{ or } h_{2C}^i = 0. \quad (48)$$

- (iii) Reformulate the year-one households' problem using the value function from the previous part.

Answer:

$$\max_{m_1^i, \{h_{1o}^i\}, s} u(m_1^i, 1 - h_{1C}^i - h_{1N}^i) + \beta V(s, x(h_{1N}^i)) \quad (49)$$

$$\text{s.t. } p_1 m_1^i + s = w_{1C} h_{1C}^i + w_{1N} h_{1N}^i + \frac{\Pi}{|I|}, \quad (50)$$

$$\text{and either } h_{1N}^i = 0 \text{ or } h_{1C}^i = 0. \quad (51)$$

- (iv) What is the marginal value of nursing experience if the individual finds it optimal to do cleaning in the second year?

Answer: Zero. By the envelope theorem,

$$\frac{\partial V_i(s, x)}{\partial x} = \lambda w_{2N} h_{2N}^i, \quad (52)$$

where λ is the Lagrange multiplier for the budget constraint. If $h_{2N}^i = 0$, then the right side simplifies to 0.

- (v) Argue informally that nurses have lower wages than cleaners in the first year.

Answer: Since some individuals choose each profession, all individuals are indifferent between being a cleaner and a nurse. Since nurses have a benefit (in the form of experience) in addition to wages, their wages must be lower in the first year.

- (vi) Are competitive equilibria Pareto efficient in this economy? (Hint: list all the differences from pure-exchange economies where we proved the first-welfare theorem, and informally discuss whether these are important.)

Answer: Yes. The major differences are:

- (a) **Production.** But home-production is equivalent.
- (b) **Experience.** This is just another form of production.
- (c) **Specialisation.** Individuals can only work in one occupation at a time. But this does not affect any part of the proof of the first welfare theorem. (The budget constraints can still be summed. Thus, we can show that a Pareto-improving allocation is worth more at market prices, and is therefore infeasible.)

(vii) * Is the excess demand function continuous?

Answer: No. At equilibrium prices, all households are indifferent between the two occupations. If the wage of cleaners increases slightly, then all households strictly prefer to specialise in cleaning, so there is a downwards jump in the excess demand of cleaners.

(viii) ** Is the household's feasible choice set compact, assuming all prices are strictly greater than zero?

Answer: Yes. It is closed because it is the intersection of these two closed sets:

- Affordable allocations (because the budget constraint is continuous).
- The set of allocations involving at most one occupation.

It is bounded, because the number of working hours is limited, so the household's wealth is limited.

15: Micro 1, May 2015

Suppose there are two schools that hire workers to teach. One school is twice as productive as the other – i.e. for the same amount of input, it produces double the output. Households supply labour and consume education.

- (i) Write down a competitive model of this economy.

Comment. The most common mistake is getting confused about how many markets there are. The most straightforward approach is to assume there is a single labour market and a single education market. An alternative approach is to assume that these markets are separate, but that households value both types of education and labour/leisure equally. The households' first-order conditions would then imply that wages are equal in both markets, and education prices are equal in both markets.

Answer: Households. Hours h , wages w , education e , price of education p , utility $u(e, 1 - h)$, school profits π_g and π_b (see below), n households. Household's problem is:

$$\begin{aligned} \max_{e,h} u(e, 1 - h) \\ \text{s.t. } pe = wh + \frac{\pi_g + \pi_b}{n}. \end{aligned}$$

Schools. School $s \in \{g, b\}$ has productivity factor $A_g = 2$ or $A_b = 1$, producing $A_s f(H)$ units of education from H hours of labour. The profit function of school s is

$$\pi_s(p, w) = \max_{H_s} pA_s f(H_s) - wH_s.$$

Equilibrium. $(h^*, e^*, H_g^*, H_b^*, p^*, w^*)$ is an equilibrium if these choices are optimal for each decision maker (as defined above), and markets clear, i.e.

$$\begin{aligned} nh^* &= H_g^* + H_b^* \\ ne^* &= A_g f(H_g^*) + A_b f(H_b^*). \end{aligned}$$

- (ii) Suppose at prevailing prices, there is excess supply of teachers. What does this imply about the supply of education?

Answer. By Walras' law, if there is excess supply in one market (of labour), then there is excess demand in another market. Since education is the only other market, we conclude there is excess demand for education.

- (iii) Prove that the “good” (more productive) school hires more teachers than the “bad” school.

Answer. The school first-order condition is

$$pA_s f'(H_s) = w,$$

which can be rearranged to

$$f'(H_s) = \frac{w}{A_s p}.$$

Since $A_g > A_b$, the right side is smaller for the good school than the bad school. By decreasing marginal productivity, we conclude that $H_g > H_b$ in every equilibrium.

- (iv) Prove that if wages increase, then schools provide less education.

Answer. By the envelope theorem,

$$\frac{\partial \pi_s(p, w)}{\partial w} = -H_s(p, w).$$

Now, π_s is the upper envelope of linear functions, so it is convex. Therefore the left side of the equation is increasing in w . It follows that $H_s(p, w)$ is decreasing in w . Total output

$$A_s f(H_s(p, w))$$

is therefore decreasing in w .

- (v) Suppose that the government imposes lump-sum taxes on half of the population, and transfers these to the other half equally. Moreover suppose that education and leisure are normal goods, and that this policy causes real wages to increase. What happens to each household's education choices? Hint: the Slutsky equation is:

$$\underbrace{\frac{\partial x_i(p, m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p, u)}{\partial p_j} \right]_{u=v(p, m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p, m)}_{\text{wealth lost}} \frac{\partial x_i(p, m)}{\partial m}}_{\text{income effect}}. \quad (53)$$

Comment. Most students struggled with this question, and overlooked that the previous part (iv) is a key ingredient. The Slutsky equation tells us about how individuals react, but the firm side of the market is also important for determining equilibrium outcomes.

Answer. By the previous part, schools supply less education, and demand less labour when real wages increase. Therefore, the total demand for education decreases.

Since real wages increased, the price of education (relative to wages) decreased. Therefore, the subsidised households have two changes to their budget constraint: the lump-sum transfer, and a price decrease of education. The first change increases wealth; this is a pure income effect which leads these households to demand more education. The second change is a price decrease in education; since education is a normal good (and hence not a Giffen good), this change leads households to consume (weakly) more education. The net effect of these changes is: the subsidised households demand more education.

Since the total demand for education decreases, the taxed households demand less education.

- (vi) * In class, to prove the existence of an equilibrium, we constructed a continuous function and proved that it has a fixed point. Since we only need to consider one price in this economy (why?), this function effectively maps from \mathbb{R} to \mathbb{R} . Describe mathematically, and sketch (i.e. draw) this function.

Answer. Since prices are relative, we can always normalise prices to sum to one. Therefore, we only need to think about one price – e.g. wages, w , since the other price is just $p = 1 - w$. By Walras' law, a wage of w forms an equilibrium if and only if the labour market clears at wage w .

Let $z_e(w)$ and $z_h(w)$ be the excess demand for education and labour, respectively. Let $Z_e(w) = \min\{z_e(w), 1\}$ and $Z_h(w) = \min\{z_h(w), 1\}$ be the truncated excess demand functions. (These are relevant when $w = 0$, which we must accommodate.)

Let $a_h(w) = \max\{0, Z_h(w)\}$ and $a_e(w) = \max\{0, Z_e(w)\}$ be the price adjustments for wages and education, respectively.

Consider the function

$$\begin{aligned} f(w) &= \frac{w + a_h(w)}{w + a_h(w) + (1 - w) + a_e(w)} \\ &= \frac{w + a_h(w)}{1 + a_h(w) + a_e(w)}. \end{aligned}$$

This function $f : [0, 1] \rightarrow [0, 1]$ is continuous. Moreover w^* is an equilibrium price if and only if w^* is a fixed point of f .

A sample graph is not included in these solutions.

- (vii) ** Let (X, d) be any metric space. Prove that if $f, g : X \rightarrow \mathbb{R}$ are continuous, then $h(x) = \max\{f(x), g(x)\}$ is also continuous. Hint: you may assume a similar result holds for addition and subtraction.

Answer. (Note: this probably isn't the simplest possible proof...)

Recall that a function $\phi : X \rightarrow Y$ is continuous if for every closed set $U \subseteq Y$, the set $\phi^{-1}(U) \subseteq X$ is closed.

We can cut X into two sets:

$$\begin{aligned} X_f &= \{x \in X : f(x) \geq g(x)\} \\ X_g &= \{x \in X : g(x) \geq f(x)\}. \end{aligned}$$

Note that X_f and X_g are closed in (X, d) . (For example, $X_f = \Delta^{-1}(\mathbb{R}_+)$, where $\Delta(x) = f(x) - g(x)$.)

Since $X = X_f \cup X_g$, we can write

$$h^{-1}(U) = [h^{-1}(U) \cap X_f] \cup [h^{-1}(U) \cap X_g] \tag{54}$$

$$= [f^{-1}(U) \cap X_f] \cup [g^{-1}(U) \cap X_g]. \tag{55}$$

Since f is continuous, $f^{-1}(U)$ is closed. Moreover, the intersections of two closed sets is closed, so $[f^{-1}(U) \cap X_f]$ is closed. Similarly, the second set on the right side is closed. The union of two closed sets is closed. We conclude that $h^{-1}(U)$ is closed. Since this logic works for any closed set U , we have established that h is continuous.

16: Micro 1, December 2015

Consider a two-generation economy in which both generations consume fish in both time periods. However, the old generation can only work in the first period and the young can only work in the second period. A fishing firm hires workers in each period to catch fish, and a storage firm hires workers to freeze fish in the first time period, and to defrost fish in the second period. Defrosted and fresh fish are perfect substitutes.

(i) Write down a competitive model of the intergenerational fishing economy.

Comment. Many students struggle to formulate the storage firm's problem correctly. For example, many students did not require the storage firm to purchase fresh fish from the fishing firm.

Answer: Let $n = n^y + n^o$ be the total population, consisting of n^y young and n^o old.

Young households. Buys fish x_t^y in time t at price p_t , works h_2^y hours in period 2 at wages w_2 , receives a share of the firms' profits $\Pi + \tilde{\Pi}$, gets utility $u^y(x_1^y, x_2^y, h_2^y)$ by:

$$\begin{aligned} \max_{x_1^y, x_2^y, h_2^y} & u^y(x_1^y, x_2^y, h_2^y) \\ \text{s.t.} & p_1 x_1^y + p_2 x_2^y = w_2 h_2^y + (\Pi + \tilde{\Pi})/n \end{aligned}$$

Old households. Similarly,

$$\begin{aligned} \max_{x_1^o, x_2^o, h_1^o} & u^o(x_1^o, x_2^o, h_1^o) \\ \text{s.t.} & p_1 x_1^o + p_2 x_2^o = w_1 h_1^o + (\Pi + \tilde{\Pi})/n. \end{aligned}$$

Fishing firm. Produces $f(H_t)$ fish from H_t hours of labour. Profit function:

$$\Pi(p_1, p_2, w_1, w_2) = \max_{H_1, H_2} p_1 f(H_1) + p_2 f(H_2) - w_1 H_1 - w_2 H_2.$$

Freezing firm. Produces $\tilde{f}(\tilde{X}_1, \tilde{H}_1, \tilde{H}_2)$ of unspoiled fish from \tilde{H}_t hours of labour in period t and \tilde{X}_1 fresh fish. Profit function:

$$\tilde{\Pi}(p_1, p_2, w_1, w_2) = \max_{\tilde{X}_1, \tilde{H}_1, \tilde{H}_2} p_2 \tilde{f}(\tilde{X}_1, \tilde{H}_1, \tilde{H}_2) - p_1 \tilde{X}_1 - w_1 \tilde{H}_1 - w_2 \tilde{H}_2.$$

Equilibrium. An allocation $(x_1^y, x_2^y, h_2^y, x_1^o, x_2^o, h_1^o, H_1, H_2, \tilde{H}_1, \tilde{H}_2)$ and prices (p_1, p_2, w_1, w_2) form an equilibrium if these choices solve the households' and firms' problems above, and markets clear:

$$\begin{aligned} n^o h_1^o &= H_1 + \tilde{H}_1 \\ n^y h_2^y &= H_2 + \tilde{H}_2 \\ n^y x_1^y + n^o x_1^o + \tilde{X}_1 &= f(H_1) \\ n^y x_2^y + n^o x_2^o &= f(H_2) + \tilde{f}(\tilde{X}_1, \tilde{H}_1, \tilde{H}_2). \end{aligned}$$

- (ii) Is it possible to normalise real wages in the first period to 1?

Answer. No. The real wage in the first period is w_1/p_1 . If we multiply all prices by α , then the real wage is unchanged.

- (iii) Show that if the price of fish in the second period increases, the storage firm sells more fish.

Answer. By the envelope theorem,

$$\begin{aligned} & \frac{\partial \tilde{\Pi}(p_1, p_2, w_1, w_2)}{\partial p_2} \\ &= \tilde{f}(\tilde{X}_1(p_1, p_2, w_1, w_2), \tilde{H}_1(p_1, p_2, w_1, w_2), \tilde{H}_2(p_1, p_2, w_1, w_2)) \\ &= \tilde{X}_2(p_1, p_2, w_1, w_2), \end{aligned}$$

where $\tilde{X}_2(p_1, p_2, w_1, w_2)$ is the optimal supply function.

Since $\tilde{\Pi}$ is the upper envelope of linear functions (one linear function for each production plan), it is convex. This means the left side of the equation above is increasing in p_2 .

It follows that the right side of the equation – supply of fish in period two – is increasing in price p_2 .

- (iv) The government is worried about intergenerational inequality, i.e. that the young will receive lower real wages than the old. It proposes a lump-sum tax on the old and transfer to the young. Show if leisure is a normal good, then this causes at least some prices to change in the new equilibrium.

Comment. Most students are able to grasp the main intuition, but have difficulty writing a logical argument. The easiest way to formulate the answer is to do a proof by contradiction. “Suppose for the sake of argument, that no prices changed. Then, some impossible things would happen, so we can rule this out.”

Answer. If the prices were the same, then the firms would choose the same production plans. This means the young would work the same amount, despite having more wealth (from transfers). This violates the assumption that leisure is a normal good.

- (v) Suppose it is only possible to store whole fish. Are all equilibria Pareto efficient?

Comment. This question requires a discussion of the proof of the first welfare theorem. Specifically, does the proof rely on divisibility?

Answer. Yes, the proof of the first welfare theorem does not depend on divisibility. The main logic is that if there were a Pareto-dominating allocation, then it would have a higher market value, and therefore be infeasible.

- (vi) * Suppose households can home-produce fish storage. Give an example of how this might lead household preferences to be time-inseparable.

Answer. The household might prefer not to buy fish tomorrow if it has fish stored from today. Specifically, consider the following four *market* choices of (x_1, h_1, x_2) :

$$a = (1, 1, 0),$$

$$b = (1, 2, 1),$$

$$c = (3, 1, 0),$$

$$d = (3, 2, 1).$$

The household might prefer $b \succ a$ and $c \succ d$, which violates time-separability.

- (vii) ** Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that if $f : X \rightarrow Y$ is continuous and X is compact in (X, d_X) , then $f(X)$ is compact in (Y, d_Y) .

Answer. We need to show that if $y_n \in f(X)$ is a sequence, then y_n has a convergent subsequence $y'_n \rightarrow_Y y'$.

Since each $y_n \in f(X)$, we know that there exists some $x_n \in X$ such that $y_n = f(x_n)$. Since X is compact, x_n has a convergent subsequence, $x'_n \rightarrow_X x'$. Let $y'_n = f(x'_n)$. Observe that y'_n is a subsequence of y_n .

Since f is continuous, $f(x'_n) \rightarrow_Y f(x')$, which means that $y'_n \rightarrow_Y f(x')$. We conclude that y'_n is a convergent subsequence of y_n , as required.

17: Micro 1, December 2015

Suppose that there are two time periods, and two seasons – summer and winter. There are about ten times as many people in the northern hemisphere than the southern hemisphere. This means that in both periods, an unequal fraction of people experience summer and winter. People prefer to work less and consume more in summer. A firm hires workers to produce a consumption good. It operates in both periods.

- (i) Write down a competitive equilibrium model of seasons and hemispheres.

Comment. The main difficulty is capturing the differences between the Northern and Southern hemispheres. Many students confuse seasons and time – seasons are of course related to time, but they are not the same thing.

Answer: Let $n = n^N + n^S$ be the total population, consisting of n^N northern and n^S southern households. There are two periods $t \in \{1, 2\}$. In the first period, it is summer in the south, and winter in the north.

Households. A household in location $\ell \in \{N, S\}$ has a discount rate of β^ℓ that depends on their location. We assume that $\beta^S < \beta^N$, which reflects the south's preference for higher consumption in the first period, etc.

Households consume $c_{\ell t}$ at price p_t , work $h_{\ell t}$ hours at wage w_t , which gives per-period utility $u(c_{\ell t}, h_{\ell t})$. Households receive dividends from firms' profits, Π . The household solves

$$\begin{aligned} \max_{\{c_{\ell t}, h_{\ell t}\}_{t=1}^2} & u(c_{\ell 1}, h_{\ell 1}) + \beta^\ell u(c_{\ell 2}, h_{\ell 2}) \\ \text{s.t.} & p_1 c_{\ell 1} + p_2 c_{\ell 2} = w_1 h_{\ell 1} + w_2 h_{\ell 2} + \pi/n. \end{aligned}$$

Firm. A single firm hire H_t hours of labour and produces $f(H_t)$ units of the consumption good in each period. Their profits are

$$\Pi(p_1, p_2, w_1, w_2) = \max_{H_1, H_2} p_1 f(H_1) + p_2 f(H_2) - w_1 H_1 - w_2 H_2.$$

Equilibrium. An allocation $(\{c_{\ell t}, h_{\ell t}\}_{t \in \{1, 2\}, \ell \in \{N, S\}}, H_1, H_2)$ and prices (p_1, p_2, w_1, w_2) form an equilibrium if these choices solve the households' and firms' problems above, and markets clear:

$$\begin{aligned} n^N h_{N1} + n^S h_{S1} &= H_1 \\ n^N h_{N2} + n^S h_{S2} &= H_2 \\ n^N c_{N1} + n^S c_{S1} &= f(H_1) \\ n^N c_{N2} + n^S c_{S2} &= f(H_2). \end{aligned}$$

- (ii) Suppose the market value of excess demand in all markets in the first time period is positive. Does this mean that there must be excess supply in a market in another time period?

Comment. This question is about Walras law, but a bit different from my usual questions. It's important to remember the big ideas behind all of the proofs – in this case “add up the households’ budget constraints”.

Answer. Yes. By Walras law, the market value of excess demand across the entire economy is 0. This means there must be some excess supply in other markets to cancel out the excess demand in the markets in the first period.

- (iii) Using dynamic programming, reformulate the households’ problems using net borrowing/lending as a state variable. That is, if this state variable is a positive number for period 1, then the household consumes more than its wages in period 1. The Bellman equation should bury the specifics about consumption or labour decisions in *both* periods.

Answer: Let $m_{\ell t}$ be the net resources devoted to period t by households in hemisphere ℓ . The households’ indirect utility function can be reformulated as:

$$V_{\ell}(p_1, p_2, w_1, w_2) = \max_{m_{\ell 1}, m_{\ell 2}} v(m_{\ell 1}; p_1, w_1) + \beta^{\ell} v(m_{\ell 2}; p_2, w_2)$$

$$\text{s.t. } m_{\ell 1} + m_{\ell 2} = \pi/n,$$

where

$$v(m, p, w) = \max_{c, h} u(c, h)$$

$$\text{s.t. } pc = wh + m.$$

- (iv) Show that households have a decreasing marginal value of net borrowing.

Answer: It suffices to show that $v(\cdot, p, w)$ is concave.

Suppose that u is concave. Suppose (c, h) is optimal for m , and (c', h') is optimal for m' . Then for any $\alpha \in (0, 1)$,

$$v(\alpha m + (1 - \alpha)m')$$

$$\geq u(\alpha c + (1 - \alpha)c', \alpha h + (1 - \alpha)h') \quad \text{since this is affordable,}$$

$$\geq \alpha u(c, h) + (1 - \alpha)u(c', h') \quad \text{since } u \text{ is concave,}$$

$$= \alpha v(m) + (1 - \alpha)v(m').$$

- (v) Show that households do more net borrowing (or less net lending) in summer than winter. *Hint: treat “how ‘northern’ a household is” as a state variable.*

Answer: Consider the value function

$$V(\beta, p_1, p_2, w_1, w_2) = \max_{m_1, m_2} v(m_1; p_1, w_1) + \beta v(m_2; p_2, w_2)$$

$$\text{s.t. } m_1 + m_2 = \pi/n.$$

This function is convex in β , because it is the upper envelope of a set of linear functions – one for each (m_1, m_2) choice. By the envelope theorem,

$$\frac{\partial V(\beta, p_1, p_2, w_1, w_2)}{\partial \beta} = v(m_2(\beta, p_1, p_2, w_1, w_2); p_2, w_2).$$

Since the left side is increasing in β , it follows that the right side is also increasing in β . Since v is increasing in resources m_2 , it follows that the optimal policy $m_2(\beta, p_1, p_2, w_1, w_2)$ is increasing in β .

This means that southern households (low β) have low net borrowing m_2 in the second period (winter), while northern households (high β) have high net borrowing m_2 in the second period (summer). The reverse is true in period one, due to the budget constraint $m_1 + m_2 = \pi/n$.

Alternative Answer: The first-order condition for the optimal savings choices is:

$$v_1(m_{\ell 1}, p_1, w_1) - \beta^\ell v_1(\pi/n - m_{\ell 1}, p_2, w_2) = 0.$$

Let $m_1 = \phi(\beta)$ be the function that is implicitly defined by this equation, i.e. that gives the relationship between discounting and the optimal amount of resources to devote to the first period. By the implicit function theorem,

$$\phi'(\beta) = -\frac{-v_1(\pi/n - m_1, p_2, w_2)}{v_{11}(m_1, p_1, w_1) + \beta v_{11}(\pi/n - m_1, p_2, w_2)}$$

Now, $v_1 > 0$ and $v_{11} < 0$ (from the previous part), so we conclude that $\phi'(\beta) < 0$. Since we assumed that $\beta^S > \beta^N$, we conclude that $m_{S1} > m_{N1}$.

This means that southern households (low β) have high net borrowing m_1 in the first period (winter), while northern households (high β) have low net borrowing m_1 in the first period (summer). The reverse is true in period two, due to the budget constraint $m_1 + m_2 = \pi/n$.

- (vi) The United Nations is worried that because of the population imbalance, the seasons create global inequality. They propose achieving equality by requiring everyone to work the same hours during summer and winter. Is it possible to design a lump-sum tax scheme that implements such an allocation? *Hint: assume that leisure is a normal good.*

Comment: Most students don't realise that the proposed allocation of resources is inefficient, so the second welfare theorem is inapplicable.

Answer: No, this is impossible. Any lump-sum tax scheme would not alter the conclusion from above that northern and southern households behave differently in terms of net borrowing/lending in the two time-periods. Since leisure is a normal good, they will still work different hours, as they have different effective income in each period and face the same prices as each other.

Since the second welfare theorem's conclusion does not hold, we conclude that its premise is false. That is, we conclude that the United Nations' target allocation is inefficient.

- (vii) ** Prove that the boundary ∂A of any set A is closed.

Answer. We would like to show that if $x_n \in \partial A$ is a sequence and $x_n \rightarrow x^*$ then $x^* \in \partial A$.

Let $\varepsilon_n = d(x_n, x^*)$; note that $\varepsilon_n \rightarrow 0$. By taking an appropriate subsequence, we may assume without loss of generality that ε_n is decreasing.

Since $x_n \in \partial A$, there exists two sequences, $(a_n)_m \in A$ and $(b_n)_m \notin A$, both of which converge to x_n . There exists subsequences $(a'_n)_m$ and $(b'_n)_m$ such that $d((a'_n)_m, x_n) < \varepsilon_m$ and $d((b'_n)_m, x_n) < \varepsilon_m$.

Let $c_n = (a'_n)_n$ and $d_n = (b'_n)_n$. By the triangle inequality,

$$d(c_n, x^*) \leq d(c_n, x_n) + d(x_n, x^*).$$

I constructed these sequences so that $d(c_n, x_n) = d((a'_n)_n, x_n) < \varepsilon_n$, and $d(x_n, x^*) = \varepsilon_n$. I conclude that

$$d(c_n, x^*) < 2\varepsilon_n$$

and hence $c_n \rightarrow x^*$. Similarly, $d_n \rightarrow x^*$. Since $c_n \in \partial A$ and $d_n \notin \partial A$, it follows that $x^* \in \partial A$.

Alternative Answer. First, notice that $\partial A = \text{cl}(A) \cap \text{cl}(A^c)$, because $\text{cl}(A)$ is the set of points that can be reached by taking the limit of a sequence inside A , and $\text{cl}(A^c)$ is the set of points that can be reached by taking the limit of a sequence of points outside of A .

Now, the closure of any set is closed, so ∂A is the intersection of two closed sets. Therefore, ∂A is closed.

18: Micro 1, May 2016

Scotland has two major cities, Glasgow and Edinburgh. Suppose that each city has an identical stock of buildings. Workers prefer to consume more buildings, and only benefit from housing located in the city that they choose to work in. There is an electronics factory in each city, that uses labour and buildings to produce electronics. The Glasgow factory is $z > 1$ times as productive as the Edinburgh factory (given the same inputs). To summarise, workers supply labour to factories, consume housing services in their own city, and consume electronics.

- (i) Write down a competitive model of the Scottish housing and electronics economy.

Answer: Let n be the population of Scotland, and \bar{B} be the building stock in each city $c \in C = \{\text{Edin, Glas}\}$.

Workers. Worker i consumes e_i electronics, $1 - h_i$ leisure, b_i buildings in city c_i . The price of electronics is p , the wage in city c is w_c , and the rent in city c is r_c . The worker's utility is $u(e_i, 1 - h_i, b_i)$. The worker owns an equal share of the building stock, $2B/n$, and in the two firms, whose profits are $\Pi = \Pi_{\text{Edin}} + \Pi_{\text{Glas}}$. The utility maximisation problem is:

$$\begin{aligned} \max_{c_i, e_i, h_i, b_i} \quad & u(e_i, 1 - h_i, b_i) \\ \text{s.t.} \quad & pe_i + r_{c_i}b_i = w_{c_i}h_i + (r_{\text{Edin}} + r_{\text{Glas}})\frac{B}{n} + \frac{\Pi}{n}. \end{aligned}$$

Firms. The factory in city c hires H_c workers, rents B_c buildings and produces $E_c = z_c f(H_c, B_c)$ items of electronics. The profit function is

$$\Pi_c(z_c, w_c, r_c) = \max_{H_c, B_c} pz_c f(H_c, B_c) - w_c H_c - r_c B_c.$$

Equilibrium. A price vector $(p, w_{\text{Edin}}, w_{\text{Glas}}, r_{\text{Edin}}, r_{\text{Glas}})$, a worker allocation $\{(c_i, e_i, h_i, b_i)\}_{i=1}^n$ and firm allocation $\{(H_c, B_c, E_c)\}_{c \in C}$ is an equilibrium if each worker's allocation solves the worker's problem, the firms' choices solve the firms' problems, and all markets clear, i.e.:

$$\begin{aligned} \sum_{i=1}^n e_i &= E_{\text{Edin}} + E_{\text{Glas}} \\ \sum_{i=1}^n I(c_i = \text{Edin})h_i &= H_{\text{Edin}} \\ \sum_{i=1}^n I(c_i = \text{Glas})h_i &= H_{\text{Glas}} \\ \sum_{i=1}^n I(c_i = \text{Edin})b_i + B_{\text{Edin}} &= \bar{B} \\ \sum_{i=1}^n I(c_i = \text{Glas})b_i + B_{\text{Glas}} &= \bar{B}. \end{aligned}$$

- (ii) Suppose that there were excess demand for workers and housing in Glasgow, and that the electronics market cleared. Does this imply that there would be excess supply of workers and/or housing in Edinburgh?

Answer: Yes, there would either be excess supply of workers or housing in Edinburgh. By Walras' law, if there is excess demand in one market, there is excess supply in at least another market. By process of elimination, this must either be the labour or housing market in Edinburgh.

- (iii) Prove that the Glasgow manufacturer's profit is increasing and convex in its productivity z .

Answer: The profit function in city c is

$$\Pi_c(z_c, w_c, r_c) = \max_{H_c, B_c} pz_c f(H_c, B_c) - w_c H_c - r_c B_c.$$

For each choice of (H_c, B_c) , the objective is linear in z_c . Therefore, Π_c is the upper envelope of linear functions in z_c . We conclude that Π_c is convex in z_c .

- (iv) Prove that if wages in Glasgow increase, then the Glasgow manufacturer demands fewer workers.

Answer: By the envelope theorem,

$$\frac{\partial}{\partial w_c} \Pi_c(z_c, w_c, r_c) = -H_c(z_c, w_c, r_c),$$

where $H_c(z_c, w_c, r_c)$ is firm c 's labour demand curve.

By similar reasoning as in the previous part, Π_c is convex with respect to wages w_c (and building rents r_c). This means the left side of the above equation is increasing in w_c . We conclude that $H_c(z_c, w_c, r_c)$ is decreasing in w_c .

- (v) Prove that if wages are higher in Glasgow, then rent is also higher in Glasgow.

Answer: Suppose for the sake of contradiction that wages are higher and rent is lower in Glasgow. If worker i chooses to live in Glasgow, his budget constraint is

$$(r_{\text{Edin}} + r_{\text{Glas}}) \frac{B}{n} + w_{\text{Glas}} h_i + \frac{\Pi}{n} - pe_i - r_{\text{Glas}} b_i \geq 0$$

If worker i chooses to live in Edinburgh, his budget constraint is

$$(r_{\text{Edin}} + r_{\text{Glas}}) \frac{B}{n} + w_{\text{Edin}} h_i + \frac{\Pi}{n} - pe_i - r_{\text{Edin}} b_i \geq 0$$

The difference is

$$(r_{\text{Edin}} - r_{\text{Glas}}) b_i + (w_{\text{Glas}} - w_{\text{Edin}}) h_i.$$

By assumption, this difference is greater than zero. This implies that worker i is less constrained in Glasgow than Edinburgh, and hence prefers to move to Glasgow. But since some workers live in each city, all workers must be indifferent between Edinburgh and Glasgow. This is a contradiction.

- (vi) Suppose there are several equilibria. Prove that every worker is indifferent between all equilibria.

Answer. In any equilibrium, all workers have the same utility as each other, since they have the same budget constraint and same utility function. Thus, if one worker is better off in a different equilibrium, then all workers are. But by the first welfare theorem, all equilibria are efficient. So no worker can be better off by switching to a different equilibrium.

- (vii) * Prove that there is only one equilibrium allocation of resources.

Answer. By the previous part, in every equilibrium, all workers have the same utility. Therefore, by the first welfare theorem, the equilibrium allocation solves the social planner's problem,

$$\begin{aligned}
& \max_{E, \{H_c, B_c\}, \{c_i, e_i, h_i, b_i\}_{i=1}^n} \sum_{i=1}^n u(e_i, h_i, b_i) \\
\text{s.t. } & \sum_{i=1}^n e_i = z_{\text{Glas}} f(H_{\text{Glas}}, B_{\text{Glas}}) + z_{\text{Edin}} f(H_{\text{Edin}}, B_{\text{Edin}}) \\
& \sum_{i=1}^n I(c_i = \text{Glas}) b_i + B_{\text{Glas}} = \bar{B} \\
& \sum_{i=1}^n I(c_i = \text{Edin}) b_i + B_{\text{Edin}} = \bar{B} \\
& \sum_{i=1}^n I(c_i = \text{Glas}) h_i = H_{\text{Glas}} \\
& \sum_{i=1}^n I(c_i = \text{Edin}) h_i = H_{\text{Edin}}.
\end{aligned}$$

The social planner's maximisation problem has a strictly concave objective, and a convex constraint set. Therefore, it has a unique solution. We conclude that there is only one equilibrium allocation.

- (viii) ** Prove that if f and g are continuous, then $h(x) = f(g(x))$ is continuous.

Answer. There are many ways to prove this, using the various equivalent definitions of continuity. I will use the open set definition: a function $\phi : X \rightarrow Y$ is continuous if for every open subset $A \subset Y$, the set $\phi^{-1}(A)$ is an open subset of X .

Now, suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. This means $h : X \rightarrow Z$. Now, pick any open set A that is a subset of Z . Since g is continuous, $g^{-1}(A)$ is an open subset of Y . Since f is continuous, $f^{-1}(g^{-1}(A))$ is an open set of X . Now, $h^{-1}(z) = f^{-1}(g^{-1}(z))$, so we conclude that $h^{-1}(A)$ is an open set. Therefore, h is continuous.

19: Micro 1, May 2016

According to Seixas, Robins, Attfield and Moulton (1992), coal miners have a 16% risk of developing the disease *black lung*. To keep things simple, suppose that all coal workers must retire early because of their health. Specifically suppose there are two time periods, and workers can choose to work in call centres or coal mines each period. After working in a coal mine, the worker is unable to work thereafter (in any job). However, sick retirees can still enjoy leisure as normal. A firm sells electricity, which it produces with coal miners and call centre workers. Workers supply either kind of labour and consume electricity and leisure.

- (i) Write down a competitive model of the electricity markets and the two types of labour markets.

Answer.

Households. Household $h \in H$ chooses their job $j_{ht} \in J = \{m, c\}$ in time $t \in T = \{1, 2\}$, where m is mining and c is call centre, their labour supply l_{ht} in time t , and electricity consumption e_{ht} in time t . The prices are w_{jt} and p_t respectively. These choices give utility $\sum_{t \in T} \beta^t u(e_{ht}, 1 - l_{ht})$, where β is the rate of time preference and u is a concave function. The household's problem is

$$\begin{aligned} & \max_{\{j_{ht}, e_{ht}, l_{ht}\}_t} \sum_{t \in T} \beta^t u(e_{ht}, 1 - l_{ht}) \\ \text{s.t. } & \sum_{t \in T} p_t e_{ht} = \sum_{t \in T} w_{j_{ht}t} l_{ht} + \frac{\Pi}{|H|}, \\ & I(j_{h1} = m) l_{h2} = 0, \end{aligned}$$

where Π is the firm's profits (see below).

Firm. The firm chooses the number of miners M_t and call centre workers C_t , which enables it to supply $E_t = f(M_t, C_t)$ units of electricity. Its profit function is

$$\begin{aligned} & \Pi(w_{1m}, w_{1c}, w_{2m}, w_{2c}, p_1, p_2) \\ & = \max_{M_1, C_1, M_2, C_2} p_1 f(M_1, C_1) + p_2 f(M_2, C_2) - w_{1m} M_1 - w_{2m} M_2 - w_{1c} C_1 - w_{2c} C_2. \end{aligned}$$

Equilibrium. A price vector $(w_{1m}, w_{1c}, w_{2m}, w_{2c}, p_1, p_2)$, a worker allocation $\{j_{ht}, e_{ht}, l_{ht}\}_{t,h}$ and a firm allocation $(M_1, C_1, E_1, M_2, C_2, E_2)$ form an equilibrium if each worker's allocation solves the worker's problem, the firm's choices solve the firm's problems,

and all markets clear, i.e.:

$$\begin{aligned}\sum_{h \in H_{m1}} l_{h1} &= M_1 \\ \sum_{h \in H_{m2}} l_{h2} &= M_2 \\ \sum_{h \in H_{c1}} l_{h1} &= C_1 \\ \sum_{h \in H_{c2}} l_{h2} &= C_2 \\ \sum_{h \in H} e_1 &= E_1 \\ \sum_{h \in H} e_2 &= E_2.\end{aligned}$$

where $H_{j't} = \{h \in H : j_{ht} = j'\}$ is the set of households who do job j' in period t .

- (ii) Reformulate the worker's problem with a Bellman equation, using wealth and health as state variables.

Answer. Let x denote wealth and $y \in \{0, 1\}$ denote health, where $y = 1$ denotes good health. The last period value function is:

$$\begin{aligned}V(x, y) &= \max_{j, e, l} u(e, 1 - l) \\ \text{s.t. } p_2 e &= w_{j2} l y + x.\end{aligned}$$

The household's problem can be written as

$$\begin{aligned}\max_{j, e, l, x'} u(e, 1 - l) + \beta V(x', I(j = c)) \\ \text{s.t. } p_1 e + x' &= w_{j1} l + \frac{\Pi}{|H|}.\end{aligned}$$

- (iii) Prove that in the last period, both professions receive the same wage.

Answer. Looking at the last period value function, the only difference between the jobs is the wage w_{j2} . If the wage in one profession were higher, then all workers would work in that profession. But then the market for the other profession would not clear (the firm will always demand some workers for each job, e.g. if production is impossible without some of each).

- (iv) Prove that the worker has diminishing marginal value of wealth in the last period.

Answer. Since the wages in the last period are equal, the choice j is immaterial, so that

$$\begin{aligned}V(x, y) &= \max_{e, l} u(e, 1 - l) \\ \text{s.t. } p_2 e &= w_{m2} l y + x.\end{aligned}$$

Fix $y = y'$, and suppose that (e, l) are optimal at (x, y) and (e', l') are optimal at (x', y') . Then

$$\begin{aligned} & \alpha V(x, y) + (1 - \alpha)V(x', y') \\ &= \alpha u(e, 1 - l) + (1 - \alpha)u(e', 1 - l') \\ &\leq u(\alpha(e, 1 - l) + (1 - \alpha)(e', 1 - l')) \\ &\leq V(\alpha(x, y) + (1 - \alpha)(x', y')). \end{aligned}$$

Therefore, V is concave in x , so the household has a diminishing marginal value of savings, i.e. $\partial V/\partial x$ is decreasing in x .

- (v) Prove that in the first period, coal miners receive higher wages than call centre workers.

Answer. Since unhealthy workers can't earn labour income in the second period, we know that $V(x, 1) > V(x, 0)$ for all x . Thus, mining imposes a cost of $V(x, 1) - V(x, 0)$ on the worker. For the worker to be indifferent between the two jobs, the mining wage w_{m1} must be higher than the call centre wage w_{c1} .

- (vi) Suppose the government selects half of the population (e.g. those born in the first half of the year) for a reward, to be funded by lump-sum taxes on the other half of the population. Is this policy Pareto efficient?

Answer. Yes. The lump-sum transfers are equivalent to re-arranging the endowments. The first welfare theorem establishes that (regardless of the endowment) all equilibria are Pareto efficient.

- (vii) ** Consider the metric space (X, d) where $X = [0, 2]$ and $d(x, y) = |x - y|$. Prove or disprove that $A = [0, 1)$ is an open set.

Answer. A is an open set.

Recall that A is open if for every point $a \in A$, there is an open neighbourhood $N_r(a) = \{b \in X : d(a, b) < r\}$ centred at a with a radius of $r > 0$ such that $N \subseteq A$.

For any point a , we can select $r = d(a, 1) = 1 - a$. With this choice of r , we need to check that $N_r(a) \subseteq A$.

Suppose $b \in N_r(a)$. Then $b \in [0, 2]$ and $d(a, b) < 1 - a$. This leads to two possibilities: $b \in [0, a]$ or $b \in (a, 2]$. For the first possibility, since $[0, a] \subseteq A$, we conclude $b \in A$. For the second possibility, $d(a, b) = b - a < 1 - a$, so that $b < 1$ and hence $b \in A$.

20: AME, mock exam

Part A. Parts (i), (ii), (iii), and (iv) of Question 19.

Part B.

(i) Let $X = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$. What is the boundary of the set

$$A = \{(x, y) \in \mathbb{R}_+^2 : x + y \leq 1\}$$

inside the metric space (X, d_2) ?

Answer. The boundary is

$$\partial A = \{(x, 0) : x \in [0, 1]\} \cup \{(0, y) : y \in [0, 1]\}.$$

First, notice that A is closed, so a point a has the property that there is some sequence $a_n \in A$ with $a_n \rightarrow a$ if and only if $a \in A$.

Second, a point (x, y) has the property that there is some sequence $b_n \in X \setminus A$ with $b_n \rightarrow (x, y)$ if and only if either $x = 0$ or $y = 0$.

The set of points satisfying both criteria form the boundary, as described above.

(ii) Consider the sequence of functions $f_n \in CB([0, 1])$ defined by $f_n(x) = x + x/n$. Is f_n a convergent sequence in $(CB[0, 1], d_\infty)$?

Answer. Yes, $f_n \rightarrow f^*$ where $f^*(x) = x$, because $d_\infty(f_n, f^*) = d_1(f_n(1), f^*(1)) = |1 + 1/n - 1| \rightarrow 0$.

(iii) Prove that $(l_\infty([0, 1]), d_\infty)$ is not a compact metric space. (Recall that $l_\infty([0, 1])$ is the set of bounded sequences $x_n \in [0, 1]$.) *Hint: you only need to find one counterexample.*

Answer. In a compact metric space, every sequence has a convergent subsequence. Now, consider the sequence

$$(x_n)_m = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Notice that $d_\infty(x_n, x_{n'}) = 1$ for all $n \neq n'$. Therefore, x_n has no convergent subsequence. We conclude that $(l_\infty([0, 1]), d_\infty)$ is not compact.

Comment. You could also add more details:

To see that $d_\infty(x_n, x_{n'}) = 1$ for all $n \neq n'$, notice that $d_\infty(x_n, x_{n'}) = \sup_m |(x_n)_m - (x_{n'})_m|$. But $(x_n)_n = 1$ and $(x_{n'})_n = 0$, so $d_\infty(x_n, x_{n'}) = 1$ for all $n \neq n'$.

Now I show that x_n has no convergent subsequence. Even if you take a subsequence of x_n by throwing out some of the sequence, it would still have the above property. But a sequence with this property can not converge. Why? Because convergent sequences are Cauchy sequences, and therefore there would have to exist some N such that $d(x_n, x_{n'}) < 1$ for all $n, n' > N$.

- (iv) Consider any metric space (X, d) . Let $x_n, y_n, z_n \in X$ be sequences. Suppose $x_n \rightarrow x^*$ and $z_n \rightarrow x^*$. Prove that if $d(x_n, y_n) \leq d(x_n, z_n)$ for all n then $y_n \rightarrow x^*$.

Answer 1. By the triangle inequality and the assumption,

$$\begin{aligned} d(y_n, x^*) &\leq d(x_n, y_n) + d(x_n, x^*) && \text{(triangle inequality)} \\ &\leq d(x_n, z_n) + d(x_n, x^*) && \text{(assumption)} \\ &\leq d(x_n, x^*) + d(z_n, x^*) + d(x_n, x^*) && \text{(triangle inequality)} \\ &= 2d(x_n, x^*) + d(z_n, x^*). \end{aligned}$$

In class, we proved that $x_n \rightarrow x^*$ if and only if $d(x_n, x^*) \rightarrow 0$. So the right side converges to 0, and hence the left side, $d(y_n, x^*)$ converges to 0. We conclude that $y_n \rightarrow x^*$.

Answer 2. Pick any $r > 0$. We would like to show that there exist some N such that

$$d(y_n, x^*) < r \text{ for all } n > N.$$

Since $x_n \rightarrow x^*$, there is some number N_x such that

$$d(x_n, x^*) < r/3 \text{ for all } n > N_x.$$

Since $z_n \rightarrow x^*$, there is an analogous number N_z .

Let $N = \max\{N_x, N_z\}$. Then,

$$\begin{aligned} d(y_n, x^*) &\leq d(x_n, y_n) + d(x_n, x^*) && \text{(triangle inequality)} \\ &\leq d(x_n, z_n) + d(x_n, x^*) && \text{(assumption)} \\ &\leq d(x_n, x^*) + d(z_n, x^*) + d(x_n, x^*) && \text{(triangle inequality)} \\ &\leq r/3 + r/3 + r/3 \\ &= r \end{aligned}$$

for all $n > N$. We conclude that $y_n \rightarrow x^*$.

- (v) Write down a recursive Bellman equation for an infinite horizon cake-eating problem in which the size of the cake grows by $r = 0.01 \times 100\%$ every day. Prove that the Bellman operator a contraction on $(CB(\mathbb{R}), d_\infty)$. What is the degree of the contraction? (You do not need to prove that the Bellman operator is a self-map.)

Answer. An appropriate Bellman equation is:

$$\begin{aligned} V(k) &= \sup_{x, k' \geq 0} u(x) + \beta V(k') \\ &\text{s.t. } x + k' \leq k(1 + r). \end{aligned}$$

Let $F(V)(k)$ be the right side of the above, i.e. the Bellman operator. We now show that F is a contraction. Consider any two value functions $V_1, V_2 \in CB(\mathbb{R}_+)$.

Let $x_1(k)$ and $x_2(k)$ be corresponding policy functions (there might be more than one if the cake-eater is indifferent). Then:

$$\begin{aligned}
F(V_1)(k) &= u(x_1(k)) + \beta V_1(k(1+r) - x_1(k)) \\
&= u(x_1(k)) + \beta V_2(k(1+r) - x_1(k)) \\
&\quad - \beta V_2(k(1+r) - x_1(k)) + \beta V_1(k(1+r) - x_1(k)) \\
&\leq u(x_1(k)) + \beta V_2(k(1+r) - x_1(k)) + \beta d_\infty(V_1, V_2) \\
&\leq [u(x_2(k)) + \beta V_2(k(1+r) - x_1(k))] + \beta d_\infty(V_1, V_2) \\
&= F(V_2)(k) + \beta d_\infty(V_1, V_2)
\end{aligned}$$

This implies that:

$$F(V_1)(k) - F(V_2)(k) \leq \beta d_\infty(V_1, V_2) \text{ for all } k \geq 0.$$

Reversing the roles of V_1 and V_2 gives the inequality:

$$F(V_2)(k) - F(V_1)(k) \leq \beta d_\infty(V_1, V_2).$$

Together, these imply that

$$d_\infty(F(V_2), F(V_1)) \leq \beta d_\infty(V_1, V_2).$$

We conclude that F is a contraction of degree β .

- (vi) Let (X, d) be a complete metric space, and $f : X \rightarrow X$ be a continuous function. Fix any $x_0 \in X$, and consider the sequence $x_{n+1} = f(x_n)$. Prove that if x_n is a Cauchy sequence then x_n converges to a fixed-point of f .

Answer. This is an important part of the proof of Banach's fixed point theorem.

Since (X, d) is complete, x_n is a convergent sequence. Let x^* be its limit. Since f is continuous, $f(x_n) \rightarrow f(x^*)$. Since $x_{n+1} = f(x_n)$, the sequence $f(x_n)$ is a subsequence of x_n , so $f(x_n) \rightarrow x^*$. Since $f(x_n)$ converges both to x^* and $f(x^*)$, we conclude that $x^* = f(x^*)$. Therefore, x_n converges to a fixed-point of f .

- (vii) Let $X = \{f \in CB([0, 1]) : f(x) = ax \text{ for some } a \in [0, 1]\}$. Is (X, d_∞) a compact metric space?

Answer. Yes. Let $F(a) = (x \mapsto ax)$, where $F : \mathbb{R} \rightarrow CB([0, 1])$. First, F is continuous: if $a_n \rightarrow a^*$ then $d_\infty(F(a_n), F(a^*)) = \sup_{x \in [0, 1]} |a_n x - a^* x| = |a_n - a^*| \rightarrow 0$. Second, $[0, 1]$ is a compact set in (\mathbb{R}, d_2) . In class, we proved that if F is continuous and A is compact then $F(A)$ is compact. Therefore, $X = F([0, 1])$ is compact.

- (viii) Prove that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x^* = 0$.

Answer. Since $\sin(1/x) \in [-1, 1]$, we know that $-x^2 \leq f(x) \leq x^2$ for all x . By the differentiable sandwich lemma, f is differentiable at $x^* = 0$.

21: AME, December 2016

Part A

CAF (Construcciones y Auxiliar de Ferrocarriles) produces trams and replacement parts for Edinburgh Trams using labour. Suppose that for each tram used in the first year of operation, 0.2 trams worth of parts must be bought for maintenance before the tram can be used in the second year. Edinburgh Trams produces public transport services from trams and labour to households. Households supply labour to the two companies and consume transport. All households have the same preferences, and shares in all firms are shared equally among all households.

- (i) Write down a general equilibrium model of the labour, tram and transportation markets involving households, the factory, and the tram operator over a two-year period. (Hint: Pay careful attention to the depreciation of trams.)

Comment. The main difficulty students had was correctly including the depreciated stock of the first year's trams into the second year.

Answer. Household's problem. There are n households, each of which receives an equal portion of the firms' profits $\pi_{CAF} + \pi_{ET}$. Households choose working hours h_t and journeys j_t in each time period $t \in \{1, 2\}$ at prices w_t and p_t respectively. This gives the household utility $u(h_1, j_1) + \beta u(h_2, j_2)$, where u is the flow utility function. The household's problem is

$$\begin{aligned} \max_{h_1, h_2, j_1, j_2} & u(h_1, j_1) + \beta u(h_2, j_2) \\ \text{s.t.} & p_1 j_1 + p_2 j_2 = w_1 h_1 + w_2 h_2 + \pi/n. \end{aligned}$$

Edinburgh Tram's problem. Edinburgh Trams purchases Y_t units of trams at a cost of r_t in period t , so that the stock is $K_1 = Y_1$ in the first year, and $K_2 = (1 - \delta)K_1 + Y_2$ in the second year, where depreciation rate is $\delta = 0.2$. They hire H_t hours of labour and sell $J_t = f(K_t, H_t)$ journeys in year t . Edinburgh Trams' profits are

$$\begin{aligned} & \pi_{ET}(p_1, p_2; r_1, r_2, w_1, w_2) \\ & = \max_{Y_1, Y_2, H_1, H_2} p_1 f(Y_1, H_1) + p_2 f((1 - \delta)Y_1 + Y_2, H_2) - w_1 H_1 - w_2 H_2 - r_1 Y_1 - r_2 Y_2. \end{aligned}$$

CAF's problem. CAF purchases H'_t units of labour in period t to produce $Y'_t = g(H'_t)$ trams. Its profits are

$$\begin{aligned} & \pi_{CAF}(r_1, r_2; w_1, w_2) \\ & = \max_{H'_1, H'_2} r_1 g(H'_1) + r_2 g(H'_2) - w_1 H'_1 - w_2 H'_2. \end{aligned}$$

Equilibrium. A vector of prices $(r_1, r_2, p_1, p_2, w_1, w_2)$ and quantities

$$(h_1, h_2, j_1, j_2, J_1, J_2, Y_1, Y_2, H_1, H_2, Y'_1, Y'_2, H'_1, H'_2)$$

form an equilibrium if they solve the problems above, and supply equals demand in each market:

$$\begin{aligned} Y_1' &= Y_1 \\ Y_2' &= Y_2 \\ nh_1 &= H_1 + H_1' \\ nh_2 &= H_2 + H_2' \\ nj_1 &= J_1 \\ nj_2 &= J_2. \end{aligned}$$

- (ii) Write down a Bellman equation for Edinburgh Trams' decision in the first year that buries the second year choices in a value function.

Comment. The key to this question is that the trams from last year is a state variable. Even if you messed up the first part, you could still try to find a way to include it here.

Answer. Let

$$\begin{aligned} &\pi_{ET2}(K_1; p_2; r_2, w_2) \\ &= \max_{Y_2, H_2} p_2 f((1 - \delta)K_1 + Y_2, H_2) - w_2 H_2 - r_2 Y_2. \end{aligned}$$

Then the Edinburgh Trams' profit function can be rewritten with a Bellman equation:

$$\begin{aligned} &\pi_{ET}(p_1, p_2; r_1, r_2, w_1, w_2) \\ &= \max_{Y_1, H_1} p_1 f(Y_1, H_1) - w_1 H_1 - r_1 Y_1 + \pi_{ET2}(Y_1; p_2, r_2, w_2). \end{aligned}$$

- (iii) Show that Edinburgh trams' second year value function is convex in prices.

Answer. Specifically, we need to show that for each capital stock K_1 , $\pi_{ET2}(K_1; \cdot; \cdot, \cdot)$ is a convex function. Now, π_{ET2} is the upper envelope of a set of convex functions. Specifically, each choice of (Y_2, H_2) has a corresponding linear (and hence convex) function

$$(p_2; w_2, r_2) \mapsto p_2 f((1 - \delta)K_1 + Y_2, H_2) - w_2 H_2 - r_2 Y_2.$$

Since upper envelopes of convex functions are convex, we conclude that $\pi_{ET2}(K_1; \cdot; \cdot, \cdot)$ is a convex function.

- (iv) Show that if the price of trams increases in the second year, then Edinburgh Trams buys fewer trams in the second year.

Answer. By the envelope theorem,

$$\frac{\partial \pi_{ET2}}{\partial r_2} = -Y_2(K_1; p_2; w_2, r_2).$$

We established above that π_{ET2} is convex in r_2 , so the left side is increasing in r_2 . It follows that the right side is increasing in r_2 , i.e. that if the price of trams increases, then Edinburgh Trams buys fewer trams in the second year.

Part B

- (i) Let (X, d) be a metric space and let $A \subseteq X$. Prove that the boundary of A is a closed set.

Comment. This is a hard question. However, it appeared in a previous exam (question 17.vii), so I was surprised that few students answered it correctly. Most students made fundamental mistakes when trying to answer this. For example, students might write that if $a \in \partial A$, then $a \in A$ (not true) or if $a_n \in A$ and $a_n \rightarrow a^*$ then $a^* \in \partial A$ (also not true).

Another common mistake was that student wrote that if $x \in \partial A$, then $\{x\}$ is a closed set (correct) and therefore the union of all of these sets is closed (incorrect). A union of a *finite* collection of closed sets is closed, but not an infinite set. For example, $\cup_n [0, (n-1)/n] = [0, 1)$ is not closed.

Answer. A short and clever answer is available in (17.vii). Here is a less creative solution.

Suppose $x_n \in \partial A$ is a convergent sequence with $x_n \rightarrow x^*$. We want to show that $x^* \in \partial A$. Specifically, we want to show that there are two sequences, namely:

- $a_n \in A$ with $a_n \rightarrow x^*$ and
- $b_n \in X \setminus A$ with $b_n \rightarrow x^*$.

Without loss of generality, assume that $d(x_n, x^*) < 1/n$. Fix any n . Since $x_n \in \partial A$, there is a sequence $\hat{a}_m \in A$ with $\hat{a}_m \rightarrow x_n$. Therefore, there exists some $a_n \in A$ such that $d(a_n, x_n) \leq 1/n$.

Now, by the triangle inequality, $d(a_n, x^*) \leq d(a_n, x_n) + d(x_n, x^*) = 1/n + 1/n \rightarrow 0$. Therefore, $a_n \rightarrow x^*$.

A similar argument establishes that there exists some $b_n \in X \setminus A$ such that $d(b_n, x_n) \leq 1/n$ hence $b_n \rightarrow x^*$.

- (ii) Suppose (X, d) is a compact metric space. Prove that if $A \subseteq X$ is a closed set, then A is a compact set.

Comment. Most students answered this question well. However, many students made mistakes. For example, students wrote that since (X, d) is compact, the limit of $a_n \in A$ has to lie in X . This reflects two misunderstandings: (1) that the definition of “limit” only makes sense if the limit is inside the metric space’s point set X , and (2) compactness only implies that a_n has a convergent subsequence; a_n itself need not be convergent.

Similarly, some students wrote that since X is compact, every subsequence of $a_n \in X$ is convergent (not true). Of course, compactness only requires that *at least one* subsequence of a_n is convergent, not *every* subsequence. If compactness required that every subsequence of a_n be convergent, then this would imply a_n itself be a convergent sequence, since a_n is a subsequence of itself.

Answer. This is an alternative to the proof given in the lecture notes.

Suppose $a_n \in A$ is a sequence. Since $a_n \in X$, it has a convergent subsequence $b_n \in A$ with $b_n \rightarrow b^*$. Moreover, since A is closed, $b^* \in A$. Thus, we have shown that a_n has a convergent subsequence whose limit lies in A . We conclude that A is compact.

- (iii) Let (A, d) be a compact metric space. Consider an optimisation problem:

$$\max_{a \in A} u(a),$$

where $u : A \rightarrow \mathbb{R}$ is continuous. Prove that the set of optimal choices,

$$A^* = \{a \in A : u(a) \geq u(a') \text{ for all } a' \in A\}$$

is compact. *Hint: use the previous question.*

Comment. My impression here is that most people had a good idea about how to prove this, but were unable to express their idea because of difficulties with mathematical notation. There are two tricks to take notice of in my answer. First, I pick out one optimal choice, and give it a name – a^* . Giving things a name is very helpful, because it means you can refer back to it in a precise way later on. Similarly, I pick out the optimal utility level, and give that a name too – u^* . This is a general lesson: *give mathematical names to important things.*

Second, I connect the set of optimal choices A^* to the utility function u via the formula $A^* = u^{-1}(\{u^*\})$. This is made much simpler because of my first trick.

Answer. By the Weierstrass theorem, there is at least one optimal choice, a^* , which gives utility $u^* = u(a^*)$. The set of all optimal choices is $A^* = u^{-1}(\{u^*\})$. Since u is continuous and $\{u^*\}$ is a closed set, it follows that A^* is a closed set. The previous question established that closed subsets of compact metric spaces are compact. Since $A^* \subseteq X$ is closed and X is compact, we conclude that A^* is compact.

- (iv) Prove that $(CB(\mathbb{R}), d_\infty)$ is *not* a compact metric space. *Hint: you only need one counterexample.*

Comment. Students gave many creative counterexamples to this question. Complicated answers are just as valid (i.e. compelling evidence) as simple ones, but I encourage students to think about the simplest possible answers.

Answer. The sequence of functions $f_n(x) = n$ has no convergent subsequence, because $d(f_n, f_m) \geq 1$ for $n \neq m$.

- (v) Suppose that the stock of salmon in the North Sea naturally doubles every five years. Individuals enjoy eating salmon according to a discounted utility function. (a) Write down a recursive Bellman equation to represent the social planner's problem over an infinite time horizon. (b) Sketch a proof that the social value of the stock of salmon is a continuous function. (You do not need to prove that the Bellman operator is a contraction, or prove the principle of optimality.)

Comment. Many students did not understand what *recursive* means – it means that it's the same value function on both sides, i.e. V and V , not V_1 and V_2 . Most students did not think of using Banach's fixed point theorem for part (b).

Answer. Let k be the stock of salmon, x be salmon consumption, $1 + r$ be the rate of natural growth, β the discount rate, $u(x)$ be the flow value of consuming Salmon, and $V(k)$ be the social value of salmon,

$$V(k) = \sup_{x, k'} u(x) + \beta V(k'(1 + r))$$

$$\text{s.t. } x + k' = k.$$

The corresponding Bellman operator,

$$F(V)(k) = \sup_{x, k'} u(x) + \beta V(k'(1 + r))$$

$$\text{s.t. } x + k' = k$$

is a contraction on the complete metric space, $(CB(\mathbb{R}_+), d_\infty)$. Therefore F has a unique fixed point, V , that is a continuous and bounded function. Since V is a fixed point of the Bellman operator, it solves the Bellman equation. By the principle of optimality, $V(k)$ is the social value of a salmon stock of k . We conclude that the social value of salmon stocks is a continuous function.

- (vi) Let $f : X \rightarrow X$ be a function on the metric space (X, d) . Prove that if f has two fixed points, $x^* \neq x^{**}$, then f is not a contraction.

Answer. Suppose for the sake of contradiction that f were a contraction of degree $a < 1$. Then $d(f(x^*), f(x^{**})) \leq ad(x^*, x^{**}) < d(x^*, x^{**})$. Since x^* and x^{**} are fixed points of f , we have $d(f(x^*), f(x^{**})) = d(x^*, x^{**})$. These two conclusions are contradictory.

- (vii) Let

$$u(x, y) = \frac{x + y}{1 + y^2 - \sqrt{y}},$$

where $(x, y) \in \mathbb{R}_+ \times [0, 1]$. Find a differentiable lower support function at $x = 2$ for

$$f(x) = \max_{y \in [0, 1]} u(x, y).$$

Comment. One trick here is that it's not necessary to solve for the optimal choice at $x = 2$. It suffices to prove that there is an optimal choice, and then just pick one and give it a name (I called it y^*).

Answer. Let y^* be an optimal choice at $x^* = 2$. (There is an optimal choice, since the objective is continuous and the choice set is compact.)

Consider the function $L(x) = \frac{x + y^*}{1 + (y^*)^2 - \sqrt{y^*}}$. This function is linear, and therefore differentiable. Moreover, $L(2) = f(2)$ and $L(x) = u(x, y^*) \leq \max_y u(x, y) = f(x)$ for all $x \in \mathbb{R}_+$. Therefore, L is a differentiable lower support function for f .

- (viii) Suppose that $f : \mathbb{R}_+^{N-1} \rightarrow \mathbb{R}$ is strictly concave. Prove that there is at most one solution to the profit maximisation problem,

$$\max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

where $(p, w) \in \mathbb{R}_{++}^N$.

Comment. Students often had good intuition, but could not convert that into a general proof. I suggest writing down relevant definitions and/or theorems to get started in writing down a proof.

Answer. Suppose for the sake of contradiction that $x^* \neq x^{**}$ are both solutions, so that

$$\pi^* = pf(x^*) - w \cdot x^* = pf(x^{**}) - w \cdot x^{**}.$$

Consider $\hat{x} = \frac{1}{2}(x^* + x^{**})$. Then,

$$\begin{aligned} pf(\hat{x}) - w \cdot \hat{x} &= pf\left(\frac{1}{2}(x^* + x^{**})\right) - w \cdot \frac{1}{2}(x^* + x^{**}) \\ &= pf\left(\frac{1}{2}(x^* + x^{**})\right) - \frac{1}{2}(w \cdot x^* + w \cdot x^{**}) \\ &> p\frac{1}{2}[f(x^*) + f(x^{**})] - \frac{1}{2}(w \cdot x^* + w \cdot x^{**}) \\ &= \pi^*. \end{aligned}$$

Thus, \hat{x} is a strictly better choice than x^* and x^{**} , contradicting the assumption that these are optimal choices.

22: Micro 1, December 2016

According to the Lincoln Longwool Sheep Breeders Association, the Lincoln Longwool sheep is “one of the most important breeds ever seen in our green and pleasant land.” It is a “dual-purpose” breed, meaning it yields high quality wool and meat. Suppose that sheep live for up to two years. If a sheep is killed at the end of the first year, it yields both wool and meat. If a sheep is killed at the end of the second year, it yields wool in both years and the same amount of meat. Households are endowed with sheep, and consume meat and wool each year. Households’ preferences can be represented with a discounted utility function. Farms buy sheep to produce wool and meat.

- (i) Write down a competitive model of the sheep, wool and meat markets across the two years.

Comment. Almost no students answered this question correctly. In particular, few students correctly accounted for dead and live sheep.

Answer.

Households. There are N households, which are endowed with s sheep. Let (w_t, m_t) be the household wool consumption and meat consumption in time period $t \in \{1, 2\}$, which gives the household utility

$$u(w_1, m_1) + \beta u(w_2, m_2).$$

The corresponding prices are (p_t^s, p_t^w, p_t^m) . In particular, p_t^s is the price of renting a sheep for period t . The household receives a share of the firm’s profits, π/N . The household’s utility maximisation problem is

$$\begin{aligned} \max_{(w_t, m_t)_{t \in \{1, 2\}}} & u(w_1, m_1) + \beta u(w_2, m_2) \\ \text{s.t.} & p_1^w w_1 + p_2^w w_2 + p_1^m m_1 + p_2^m m_2 = (p_1^s + p_2^s)s + \frac{\pi}{N} \end{aligned}$$

Farm. It allocates K_t (killed) sheep for meat and wool production, and L_t (live) sheep for wool production only, so that $M_t = f(K_t)$ meat is produced and $W_t = g(K_t + L_t)$ wool is produced. It needs to acquire $S_1 = K_1 + L_1$ sheep in year 1, and $S_2 = K_1 + K_2 + L_2$ in year 2. The farm’s profit function is

$$\begin{aligned} & \pi((p_t^s, p_t^w, p_t^m)_{t \in \{1, 2\}}) \\ & = \max_{K_1, K_2, L_1, L_2} p_1^m f(K_1) + p_2^m f(K_2) + p_1^w g(K_1 + L_1) + p_2^w g(K_2 + L_2) \\ & \quad - p_1^s(K_1 + L_1) - p_2^s(K_1 + K_2 + L_2). \end{aligned}$$

Equilibrium. A price vector $(p_t^s, p_t^w, p_t^m)_{t \in \{1, 2\}}$ and an allocation

$$(w_t, m_t, K_t, L_t)_{t \in \{1, 2\}}$$

constitute an equilibrium if the allocation solves the choice problems above, and markets clear:

$$\begin{aligned}
Ns &= K_1 + L_1 \\
Ns &= K_1 + K_2 + L_2 \\
Nm_1 &= f(K_1) \\
Nm_2 &= f(K_2) \\
Nw_1 &= g(K_1 + L_1) \\
Nw_2 &= g(K_2 + L_2).
\end{aligned}$$

- (ii) Prove that farms demand more sheep in the first year if the price of sheep decreases (but no other prices change).

Answer. By the envelope theorem,

$$\begin{aligned}
& \frac{\partial \pi(p_1^s, p_1^m, p_1^w, p_2^s, p_2^m, p_2^w)}{\partial p_1^s} \\
&= \left[\frac{\partial}{\partial p_1^s} \left(p_1^m f(K_1) + p_2^m f(K_2) + p_1^w g(K_1 + L_1) + p_2^w g(K_2 + L_2) \right. \right. \\
&\quad \left. \left. - p_1^s(K_1 + L_1) - p_2^s(K_1 + K_2 + L_2) \right) \right]_{\text{at optimal } L_1, K_1, L_2, K_2} \\
&= -[K_1 + L_1]_{\text{at optimal } L_1, K_1} \\
&= -S_1(p_1^s, p_1^m, p_1^w, p_2^s, p_2^m, p_2^w).
\end{aligned}$$

Now, the profit function π is a convex function, since it is the upper envelope of linear functions (one linear function for each (K_1, L_1, K_2, L_2)). Therefore, $\frac{\partial \pi}{\partial p_1^s}$ is increasing in p_1^s . Since the left side is increasing, we conclude that sheep demand S_1 is a decreasing function of the price of sheep p_1^s .

- (iii) Write down the firm's value of owning live sheep in the first and second years, making use of a Bellman equation. Prove that these are concave functions of the number of sheep.

Answer. The value of S_1 sheep in the first year is

$$\begin{aligned}
V_1(S_1; p_1^w, p_1^m, p_2^w, p_2^m) &= \max_{K_1, S_2} p_1^m f(K_1) + p_1^w g(S_1) + V_2(S_2; p_2^w, p_2^m) \\
&\text{s.t. } K_1 + S_2 = S_1.
\end{aligned}$$

The value of S_2 sheep in the second year is

$$V_2(S_2; p_2^w, p_2^m) = p_2^m f(S_2) + p_2^w g(S_2).$$

If we assume that the production functions f and g are concave in S_2 , then V_2 is the sum of two concave functions, and hence is concave in S_2 .

Similarly, the objective in the Bellman equation is jointly concave in (S_1, K_1, S_2) and the constraint is linear, so V_1 is concave in S_1 (by a Theorem from lectures).

- (iv) Find an assumption on the model parameters such that the price of sheep decreases over time.

Answer. The firm's first-order conditions with respect to live-sheep are:

$$\begin{aligned} p_1^w g'(K_1 + L_1) &= p_1^s \\ p_2^w g'(K_2 + L_2) &= p_2^s. \end{aligned}$$

By the market clearing condition and looking at endowments, note that $K_2 + L_2 = L_1$.

The household's first-order conditions with respect to wool consumption are:

$$\begin{aligned} u_w(w_1, m_1) &= \lambda p_1^w \\ \beta u_w(w_2, m_2) &= \lambda p_2^w. \end{aligned}$$

where λ is the Lagrange multiplier on the budget constraint.

Substitution gives:

$$\begin{aligned} \frac{1}{\lambda} u_w(w_1, m_1) g'(K_1 + L_1) &= p_1^s \\ \frac{1}{\lambda} \beta u_w(w_2, m_2) g'(L_1) &= p_2^s. \end{aligned}$$

Therefore $p_1^s > p_2^s$ if and only if

$$u_w(w_1, m_1) g'(K_1 + L_1) > \beta u_w(w_2, m_2) g'(L_1).$$

Now, if we assume that g has constant returns to scale, then $p_1^s > p_2^s$ is if

$$u_w(w_1, m_1) > \beta u_w(w_2, m_2).$$

If $\beta = 0$, then this inequality holds; it would also hold for small β if u_w is bounded. Specifically, if there are some number $x, y > 0$ such that $u_w(w, m) \in [x, y]$ for all (w, m) , then the left side is bigger if $\beta < \frac{x}{y}$.

Conclusion: if β is close to zero, the marginal utility of wool is bounded, and the wool production function has constant returns to scale, then the price of sheep decreases over time.

- (v) Suppose that half of the population is poor, and only has only half of the sheep endowment. Is it possible to devise a lump-sum transfer scheme that institutes equal welfare for each household?

Comment. You can answer either by citing the second welfare theorem, or by constructing the policy directly. If you apply the second welfare theorem, you have to prove that the target allocation is efficient (which can be done via the first welfare theorem).

Answer. Yes. Suppose the wealthy households are endowed with s sheep, and the poor households are endowed with $s/2$ sheep. Let $(\hat{p}_t^s, \hat{p}_t^m, \hat{p}_t^w)$ be the equilibrium prices once the tax regime is implemented. Taxing the wealthy households $(\hat{p}_1^s + \hat{p}_2^s)s/4$ and transferring this value to the poor households would give all households the same budget constraint. Therefore, all households would have the same preferences and same budget constraint, so they would have the same welfare.

- (vi) * Let $X = \mathbb{R}_+^6$. Suppose there is a continuous function $f : X \rightarrow X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate f .* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then f has a fixed point.

Comment. The question on the exam incorrectly defined X as \mathbb{R}_{++}^6 .

Answer.

- (a) Let $Y = \left\{ p \in X : \sum_{i=1}^N p_i = 1 \right\}$ and

$$g(x) = \frac{f(p)}{\sum_{i=1}^N f_i(p)}.$$

Notice that $g : Y \rightarrow Y$ is continuous (since f and the standard operations $+$ and $/$ are continuous). Next, Y is closed, bounded, and convex, so Brouwer's fixed point theorem implies that g has a fixed point p^* .

Now, let $t = \sum_{i=1}^N f_i(p^*)$. By construction, $f(p^*) = tp^*$. By property (2) $f(f(p^*)) = f(tp^*) = f(p^*)$. We conclude that $f(p^*)$ is a fixed point of f . By property (1) $f(p^*)$ is an equilibrium price vector.

- (b) This is an important part of the proof of Banach's fixed point theorem.

Since (X, d_2) is complete, p_n is a convergent sequence. Let p^* be its limit. Since f is continuous, $f(p_n) \rightarrow f(p^*)$. Since $p_{n+1} = f(p_n)$, the sequence $f(p_n)$ is a subsequence of p_n , so $f(p_n) \rightarrow p^*$. Since $f(p_n)$ converges both to p^* and $f(p^*)$, we conclude that $p^* = f(p^*)$. Therefore, p^* is a fixed point of f .

23: Micro 1, December 2016

Suppose a country consists of workers with and without university degrees. Only university graduates can design machines, but both types of worker are equally competent at operating machines. There are two firms: a machine manufacturer that hires university graduates and a clothing manufacturer that buys machines and can hire either type of worker. Workers sell labour and consume clothing and machines (for washing their clothes).

- (i) Formulate a competitive equilibrium model of the markets for both types of labour, machines and clothing. *Hint: do not assume that equilibria are symmetric.*

Comment. The original formulation of the question did not make it clear that machines are necessary to make clothes. As a result, all students rightly did not include machines in the clothing production function – it is good to keep your model as simple as possible.

Answer.

Workers' problem. Let H be the set of workers, and let $G \subseteq H$ be the set of graduates. Let (e_g^h, e_n^h) be the endowment of worker $h \in H$ of graduate hours and non-graduate hours of labour. Assume that if $h \in G$ then $(e_g^h, e_n^h) = (1, 0)$; otherwise $(e_g^h, e_n^h) = (0, 1)$. Worker h supplies (l_g^h, l_n^h) units of labour at wages (w_g, w_n) , and consumes c^h items of clothing at price p and m^h machines at price r . The worker receives a share of the firms' profits $\frac{\pi}{|H|}$, where $\pi = \pi^m(r; w_g) + \pi^c(p; w_g, w_n, r)$ is defined below. The worker's utility is $u(l_g^h + l_n^h, c^h, m^h)$, so the utility maximisation problem is

$$\begin{aligned} \max_{l_g^h, l_n^h, c^h, m^h} & u(l_g^h + l_n^h, c^h, m^h) \\ \text{s.t.} & pc^h + rm^h = w_g l_g^h + w_n l_n^h + \frac{\pi}{|H|} \text{ and } l_g^h \leq e_g^h \text{ and } l_n^h \leq e_n^h. \end{aligned}$$

Machine manufacturer's problem. The machine manufacturer hires L_g^m graduates and sells $M^m = f(L_g^m)$ machines. Its profits are

$$\pi^m(r; w_g) = \max_{L_g^m} r f(L_g^m) - w_g L_g^m.$$

Clothing manufacturer's problem. The clothing manufacturer hires L_g^c graduates and L_n^c non-graduates, and buys M^c machines to sell $C^c = g(L_g^c + L_n^c, M^c)$ units of clothing. Its profits are

$$\pi^c(p; w_g, w_n, r) = \max_{L_g^c, L_n^c, M^c} pg(L_g^c + L_n^c, M^c) - w_g L_g^c - w_n L_n^c - r M^c.$$

Equilibrium. The prices (p, r, w_g, w_n) and allocation

$$\left(\{c^h, l_g^h, l_n^h, m^h\}_{h \in H}, L^m, M^m, C^c, L_g^c, L_n^c, M^c \right)$$

form an equilibrium if the choices solve the respective problems above, and each market clears:

$$\begin{aligned}\sum_{h \in H} l_g^h &= L_g^m + L_g^c \\ \sum_{h \in H} l_n^h &= L_n^c \\ \sum_{h \in H} c^h &= C^c \\ \sum_{h \in H} m^h + M^c &= M^m.\end{aligned}$$

- (ii) Suppose at some market prices, the supply of university-educated labour exceeds demand. Does this imply that the demand for uneducated labour exceeds supply?

Answer. While Walras' law implies that there must be excess supply in one market, it need not be in the uneducated labour market.

- (iii) Suppose the two firms decide to merge into single firm. (a) Write the combined-firm's profit-maximisation problem using a Bellman equation to separate the output and input choices. (b) Does the equilibrium (or equilibria) change after the merger?

Comment. One common mistake in this part was to combine the graduate labour demand across the two activities (making machines and making clothes). Students who made this mistake effectively assumed that university graduates could spend the whole day doing both tasks simultaneously.

Another common mistake was to forget to include the wages as state variables.

Answer.

- (a) The profit function is

$$\pi(p, r, w_g, w_n) = \max_{C, M} pC + rM - e(C, M; w_g, w_n)$$

where the production cost is

$$\begin{aligned}e(C, M; w_g, w_n) &= \min_{M^c, L_g^c, L_n^c, L_g^m} w_g(L_g^c + L_g^m) + w_n L_n^c \\ \text{s.t. } f(L_g^m) &\geq M + M^c \text{ and } g(L_g^c + L_n^c, M^c) \geq C.\end{aligned}$$

- (b) No. The merged firm would make the same choices as the two separate firms if confronted with the same prices. Therefore, the set of equilibria would be unchanged.

- (iv) Prove that if the wages of uneducated workers increases, the clothing manufacturer hires fewer uneducated workers.

Answer. By the envelope theorem,

$$\frac{\partial \pi^c(p; r, w_g, w_n)}{\partial w_n} = -L_n(p; r, w_g, w_n).$$

Now, π^c is convex, as it is the upper envelope of a set of linear functions (one for each input choice (M^c, L_g^c, L_n^c)). Therefore, the left side is increasing in w_n . We conclude that $L_n(p; r, w_g, w_n)$ is decreasing in w_n , i.e. when uneducated wages increase, the clothing manufacturer's demand for uneducated labour decreases.

- (v) Prove that if the clothing manufacturer hires educated workers, then the wages paid to all workers by both firms are equal.

Comment. A common mistake was to differentiate the production function without specifying which partial derivative is relevant.

Many students did not assume that both types of workers are perfect substitutes in the clothing production function, which is important.

Many students confused the envelope theorem with first-order conditions. First-order conditions are about differentiating with respect to choice variables. The state variables of the profit function are all prices, which are not choice variables, so it makes no sense to apply the envelope theorem here. (The envelope theorem *is* useful for first-order conditions when the state variable is a choice, e.g. with production targets in the cost function.)

Answer. If the clothing manufacturer hires both types of worker, then the first-order conditions are:

$$\begin{aligned} L_g^c: pg_1(L_g^c + L_g^m, M^c) &= w_g \\ L_n^c: pg_1(L_g^c + L_g^m, M^c) &= w_n. \end{aligned}$$

Since the left sides are equal, the right sides are equal and $w_g = w_n$.

- (vi) Suppose every Pareto efficient allocation involves university graduates working for the machine manufacturer only. Is it possible to find lump-sum transfers to implement an allocation in which some university graduates work for the clothing manufacturer?

Comment. A common mistake here was to apply the second welfare theorem. But recall that this theorem only applies if the target allocation is efficient. The question implicitly states that the target allocation is inefficient.

Answer. No. By the first welfare theorem, every competitive equilibrium is efficient (no matter which endowments the households are allocated). Therefore, every competitive equilibrium with lump-sum transfers involves graduates working for the machine manufacturer only.

- (vii) * Let $X = \mathbb{R}_+^4$. Suppose there is a continuous function $f : X \rightarrow X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate f .* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then f has a fixed point.

Answer. See the answer to the last part of the previous question.

24: AME, May 2017

Part A

Until plastic bottles became popular in the 1960s, milk was sold in glass bottles that could be reused. For simplicity, assume there are two time periods. Suppose households supply labour, and buy bottled milk and empty bottles in both periods. Milk bottles from the first period become empty in the second period, and households can sell these (or buy even more). A firm uses labour to make bottles and bottled milk in both periods.

(i) Write down a competitive model of the bottled milk industry.

Comment. While most students described a self-consistent model, they did not capture the essence of the question:

- Milk producers make milk bottles out of empty bottles (and labour to make the milk).
- Households can sell their used bottles back to the milk producers.

Another common mistake was to only consider the total labour demand in each period, without thinking about how this labour is allocated between milk and bottle production. (For example, several students assumed that all workers conducted both activities simultaneously.)

Answer. We construct a representative **household** model with n identical households. Let m_t be milk consumption, b_t be empty bottle usage, and h_t be labour supply in time period $t \in \{1, 2\}$. These trade at prices p_t , r_t and w_t respectively. The household is endowed with 1 unit of labour each period, and $1/n$ shares of the firm's profit Π (see below). The household maximises its utility function, $u(m_1, b_1, h_1) + \beta u(m_2, b_2, h_2)$. The household's problem is

$$\begin{aligned} & \max_{(m_t, b_t, h_t)_{t \in \{1, 2\}}} u(m_1, b_1, h_1) + \beta u(m_2, b_2, h_2) \\ & \text{s.t. } p_1 m_1 + p_2 m_2 + r_1 b_1 = r_2(m_1 + b_1 - b_2) + w_1 h_1 + w_2 h_2 + \Pi/n. \end{aligned}$$

The **firm** allocates H_t^b and H_t^m units of labour to bottle and milk production in period t . The firm allocates B_t^m bottles for milk production in period t . The firm's bottle output is $B_t = f(H_t^b)$, and its milk output is $M_t = g(H_t^m, B_t^m)$ in period t . The firm's profit function is

$$\begin{aligned} & \Pi(p_1, p_2, r_1, r_2, w_1, w_2) \\ & = \max_{H_1^b, H_2^b, H_1^m, H_2^m, B_1^m, B_2^m} \\ & \quad p_1 g(H_1^m, B_1^m) + p_2 g(H_2^m, B_2^m) + r_1 [f(H_1^b) - B_1^m] + r_2 [f(H_2^b) - B_2^m] \\ & \quad - w_1 (H_1^m + H_1^b) - w_2 (H_2^m + H_2^b) \end{aligned}$$

An **equilibrium** consists of prices

$$(p_1, p_2, r_1, r_2, w_1, w_2)$$

and quantities

$$(m_1, m_2, h_1, h_2, b_1, b_2, H_1^b, H_2^b, H_1^m, H_2^m, B_1^m, B_2^m)$$

such that the choices solve the household and firm problems above, and all six markets clear:

$$nm_1 = M_1 \quad (56)$$

$$nm_2 = M_2 \quad (57)$$

$$nb_1 + B_1^m = B_1 \quad (58)$$

$$nb_2 + B_2^m = B_1 + B_2 \quad (59)$$

$$nh_1 = H_1^m + H_1^b \quad (60)$$

$$nh_2 = H_2^m + H_2^b. \quad (61)$$

- (ii) Reformulate the firm's problem by separating the firm's milk and bottle production decisions. *Hint: this is a bit like dynamic programming, but the "Bellman equation" has no choice variables.*

Answer. The firm could split into two firms, whose profit functions are related by the (trivial) Bellman equation

$$\Pi(p_1, p_2, r_1, r_2, w_1, w_2) = \Pi^m(p_1, p_2, r_1, r_2, w_1, w_2) + \Pi^b(r_1, r_2, w_1, w_2)$$

where

$$\begin{aligned} & \Pi^m(p_1, p_2, r_1, r_2, w_1, w_2) \\ &= \max_{H_1^m, H_2^m, B_1^m, B_2^m} \\ & \quad p_1 g(H_1^m, B_1^m) + p_2 g(H_2^m, B_2^m) - r_1 B_1^m - r_2 B_2^m - w_1 H_1^m - w_2 H_2^m \end{aligned}$$

and

$$\Pi^b(r_1, r_2, w_1, w_2) = \max_{H_1^b, H_2^b} r_1 f(H_1^b) + r_2 f(H_2^b) - w_1 H_1^b - w_2 H_2^b.$$

- (iii) Prove that the firm has an increasing marginal profit (i.e. a decreasing marginal loss) of a second-period wage increase.

Answer. Π is the upper envelope of functions that are linear in prices (one for each choice vector). Therefore, Π is a convex function of prices, and hence a convex function of second-period wages. Therefore, its derivative with respect to second period wages is an increasing function.

- (iv) Prove that the firm reacts to a second-period bottle price increase by increasing its net supply of (empty and filled) bottles.

Answer. By the envelope theorem,

$$\frac{\partial \Pi}{\partial r_2} = f(H_2^b(p_1, p_2, r_1, r_2, w_1, w_2)) - B_2^m(p_1, p_2, r_1, r_2, w_1, w_2). \quad (62)$$

The right side of this equation is the net supply of bottles. Since the left side is increasing in r_2 (see the previous question), the right side is also increasing.

Part B

Comment. Students made several common types of mistakes in this section:

- Students confused the meanings of “there exists” versus “for all”.
- Students were confused about the definition of convergence. The phrase “for all $r > 0$, there exists an N such that” means something very different from “there exists an N such that for all $r > 0$ ”.

- (i) Consider the metric space (X, d_2) where $X = [0, 1] \times \mathbb{R}$ and $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. What is the boundary of the set $A = [0, 1] \times \{0\}$ in this space?

Answer. The boundary of A is A itself.

First we show that if $(u^*, v^*) \in A$, then $(u^*, v^*) \in \partial A$. Then the sequence $(u_n, v_n) = (u^*, 1/n) \notin A$ converges to (u^*, v^*) . And the trivial sequence $(u'_n, v'_n) = (u^*, v^*) \in A$ also converges to (u^*, v^*) . Therefore, $(u^*, v^*) \in \partial A$.

Second, we show that if $(u^*, v^*) \notin A$, then $(u^*, v^*) \notin \partial A$. Since $(u^*, v^*) \notin A$, we know that $v^* \neq 0$. Let $r = |v^*|$, where $r > 0$. Since the open ball $N_r(u^*, v^*)$ does not overlap with A , no sequence in A converges to (u^*, v^*) .

- (ii) Let $X = \{f : [0, 1] \rightarrow \mathbb{R} \text{ s.t. } f \text{ is continuously differentiable}\}$ and

$$d(f, g) = d_\infty(f, g) + d_\infty(f', g'),$$

where f' and g' are the derivatives of f and g respectively, and $d_\infty(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|$. Prove (a) d is well-defined and (b) (X, d) is a metric space.

Comment. While it's possible to answer this question from first principles, the most elegant approach is to make use of the fact that $(CB([0, 1]), d_\infty)$ is a metric space. (If you are worried that this is “cheating”, you could prove this fact separately.)

Answer. (a) Checking that d is well-defined requires checking that d exists and is unique. Since $f, g \in X$, it follows that $f, g, f', g' : [0, 1] \rightarrow \mathbb{R}$ are continuous functions. It follows that $x \mapsto |f(x) - g(x)|$ and $x \mapsto |f'(x) - g'(x)|$ are continuous functions. By the Weierstrass Theorem, the maxima of these continuous functions on the compact domain $[0, 1]$ exist. Therefore, $d_\infty(f, g)$ and $d_\infty(f', g')$ exist. So d exists. Uniqueness is by construction: the supremums are unique, so their sum is unique too.

(b) We now prove that (X, d) is a metric space. We make use of the fact that (X, d_∞) is a metric space.

- $d(f, g) = 0$ if and only if $f = g$. Suppose $d(f, g) = 0$. Then $d_\infty(f, g) = 0$, and hence $f = g$.

Suppose $f = g$. Then $f' = g'$. So $d_\infty(f, g) = 0$ and $d_\infty(f', g') = 0$. We conclude $d(f, g) = 0$.

- $d(f, g) = d(g, f)$. This follows from $d_\infty(f, g) = d_\infty(g, f)$ and $d_\infty(f', g') = d_\infty(g', f')$.

- $d(f, h) \leq d(f, g) + d(g, h)$. Note that

$$d_\infty(f, h) \leq d_\infty(f, g) + d_\infty(g, h), \quad (63)$$

$$d_\infty(f', h') \leq d_\infty(f', g') + d_\infty(g', h'). \quad (64)$$

Summing the two inequalities gives the conclusion.

- (iii) Consider the metric space (X, d_1) where $X = (0, 1)$ and $d_1(x, y) = |x - y|$. Supply a counter-example to prove that (X, d_1) is *not* complete.

Comment. Several students wrote “ x_n wants to converge to x^* ” without being aware of the limitations of using informal intuitive language rather than precise mathematical language. Intuitive language has its place in mathematical writing – it is very helpful for conveying difficult ideas (and this is why I speak this way in lectures). However, it is not a substitute for being precise; it should be used in addition to, not instead of precise language.

The problem with writing “wants to converge to” is it is terminology that has not been defined. It is probably possible to come up with a definition, but that would probably defeat the advantages of informal language. This means that the phrase is ambiguous. For example, suppose x_n is a sequence inside the metric space (X, d) , but $x^* \notin X$. This means that x_n can not converge to x^* , because limits must lie inside X . The intuitive answer to this is that we can imagine a bigger metric space, (Y, d) , that somehow extends (X, d) . So “wanting to converge to x^* ” means that x_n , when considered a sequence in (Y, d) , converges to x^* . But this is still ambiguous, because there might be many different metric spaces (Y, d) that extend (X, d) in the right way.

To summarise: when you write proofs, you are very welcome to use intuitive and ambiguous language, *provided that you subsequently clarify exactly what you mean*.

Answer. We will construct a non-convergent Cauchy sequence. Let $x_n = 1/(n+1)$. In the metric space $([0, 1], d_1)$, the sequence $x_n \rightarrow 0$. Therefore, x_n is a Cauchy sequence in (X, d_1) , since the definition of Cauchy sequence is only based on the definition of the metric. Now, x_n does not converge, so (X, d_1) is not complete.

- (iv) Consider the metric space (X, d_1) , where $X \subseteq [0, 1]$ and $d_1(x, y) = |x - y|$. Suppose that $x_n \in X$ has no convergent subsequence. Prove that X is not a closed set in (\mathbb{R}, d_1) .

Comment. Most students did not realise that compactness (i.e. the Bolzano-Weierstrass Theorem) is the key to this question. Compactness is the property that ensures that all sequences in $([0, 1], d_1)$ have convergent subsequences. Because of this logical gap, students were quite creative in constructing specious arguments. For example, some students implicitly added the extra assumption that $x_n \rightarrow x^*$, where $x^* \in [0, 1]$.

Answer. Suppose for the sake of contradiction that X were a closed set subset of $[0, 1]$. Then X is a closed and bounded set, so the Bolzano-Weierstrass Theorem implies that $x_n \in X$ has a convergent subsequence. But x_n has no convergent subsequence. Therefore, the supposition that X is closed is false.

- (v) Let $x_t \in [0, 1]$ be the fraction of the population of generation t that is religious. Suppose that each subsequent generation's demographics are deterministic with $x_{t+1} = f(x_t)$, and that $x_t \rightarrow x^*$. Prove that if f is a continuous function, then x^* is a fixed point of f , i.e. x^* is a steady state.

Answer. Since f is continuous, $y_t = f(x_t)$ converges to $f(x^*)$. Now, y_t is a subsequence of x_t , so $y_t \rightarrow x^*$. Thus y_t converges to both x^* and $f(x^*)$. Since sequences can converge to only one point, we conclude that $x^* = f(x^*)$.

- (vi) Prove that $f(x) = \frac{1}{3}x^2$ is a contraction on the metric space $(X, d) = ([0, 1], d_1)$ where $d_1(x, y) = |x - y|$.

Answer.

$$d_1(f(x), f(y)) = \frac{1}{3}d_1(x^2, y^2) \quad (65)$$

$$= \frac{1}{3}|x^2 - y^2| \quad (66)$$

$$= \frac{1}{3}|(x - y)(x + y)| \quad (67)$$

$$= \frac{1}{3}d_1(x, y)|x + y| \quad (68)$$

$$\leq \frac{2}{3}d_1(x, y). \quad (69)$$

Therefore, f is a contraction of degree $\frac{2}{3}$.

- (vii) Consider a two player-game where player one and two choose $a \in [0, 1]$ and $b \in [0, 1]$ respectively. Suppose that player one and two have best response functions $f(b)$ and $g(a)$ respectively. Let $X = A \times B$ and $h : X \rightarrow X$ be defined by $h(a, b) = (f(b), g(a))$. Consider the following procedure (called iterated deletion of dominated strategies) for calculating Nash equilibria:

(a) Set $Y_1 = X$.

(b) Let $Y_{n+1} = h(Y_n)$, that is $Y_{n+1} = \{h(a, b) : (a, b) \in Y_n\}$.

(c) Report $Y_\infty = \bigcap_{n=1}^\infty Y_n$.

Prove that if h is continuous, then $Y_\infty \neq \emptyset$, i.e. that this procedure does not delete all strategies. *Hint: Apply the Cantor intersection theorem.*

Answer. We will show that each Y_n is compact, non-empty, and $Y_{n+1} \subseteq Y_n$. Then Cantor's intersection theorem establishes that Y_∞ is non-empty (and compact, but that is not relevant here).

Since h is continuous and Y_1 is non-empty and compact, it follows that $Y_2 = h(Y_1)$ is non-empty and compact. Repeating this argument establishes that each Y_n is non-empty and compact.

By assumption, $h(Y_1) \subseteq Y_1$, since best-responses must lie in X . Therefore, $h(h(Y_1)) \subseteq h(Y_1)$, and hence $Y_3 \subseteq Y_2$. Continuing this logic establishes that each $Y_{n+1} \subseteq Y_n$.

Therefore, Cantor's theorem applies.

(viii) Recall that $CB(\mathbb{R}_+)$ is the set of continuous and bounded functions with domain \mathbb{R}_+ and co-domain \mathbb{R} , whose distances can be measured with the metric

$$d_\infty(f, g) = \sup_{x \in \mathbb{R}_+} |f(x) - g(x)|.$$

Consider the following Bellman operator $\Phi : CB(\mathbb{R}_+) \rightarrow CB(\mathbb{R}_+)$, which is a contraction of degree β on $(CB(\mathbb{R}_+), d_\infty)$:

$$\begin{aligned} \Phi(V)(k) &= \sup_{c, k'} u(c) + \beta V(k') \\ \text{s.t. } c + k' &= g(k). \end{aligned}$$

(You may interpret c as consumption, k as capital, $g(k)$ as output $u(c)$ as utility, and β as the rate of time preference.) Use Banach's fixed point theorem to prove that if u and g are concave, then the fixed point of Φ is concave.

Answer. We may reformulate Φ without the constraint as

$$\Phi(V)(k) = \sup_{k'} u(g(k) - k') + \beta V(k').$$

We now show that if V is a concave function, then $\Phi(V)$ is also concave function. Let $h(k, k') = u(g(k) - k') + \beta V(k')$ be the objective function. Since $u, g, k' \mapsto -k'$ and V are concave functions it follows that h is a concave function. Now, let k'_1 and k'_2 maximise $h(k_1, \cdot)$ and $h(k_2, \cdot)$ respectively. (By the extreme value theorem, these exist.) Then,

$$\Phi(V)(tk_1 + (1-t)k_2) = \sup_{k'} h(tk_1 + (1-t)k_2, k') \quad (70)$$

$$\geq h(tk_1 + (1-t)k_2, tk'_1 + (1-t)k'_2) \quad (71)$$

$$\geq th(k_1, k'_1) + (1-t)h(k_2, k'_2) \quad (72)$$

$$= t\Phi(V)(k_1) + (1-t)\Phi(V)(k_2). \quad (73)$$

Thus, $\Phi(V)$ is concave whenever V is concave.

Let $X = \{f \in CB(\mathbb{R}) : f \text{ is concave}\}$. We have established that if we restrict Φ to X then Φ is a contraction in the metric space (X, d_∞) . In tutorials, we established that (X, d_∞) is a complete metric space. Therefore Banach's fixed point theorem establishes that Φ has a unique fixed point $V^* \in X$, i.e. there is a fixed-point of Φ that is concave.

Similarly, Banach's fixed point theorem establishes that Φ has only one fixed point in $CB(\mathbb{R}_+)$, so the only fixed point must be V^* . We conclude that the fixed point of Φ is concave.

25: Micro 1, May 2017

Consider an economy with two time-periods, in which the entire population lives for both periods. The young and old are identical, except the young have no labour endowment in the first period. They can supply up to their labour endowment and consume food in each period, and have time-separable preferences. A farm produces food from labour.

- (i) Devise a competitive model of the food and labour markets.

Answer. Households. Each household $h \in H$ belongs to either generation $h \in X$ or $h \in Y$. In period 1, each old household $h \in X$ has a first-period labour endowment of $e_{h1} = 1$; the young households' $h \in Y$ have $e_{h1} = 0$. Apart from that households are identical. They all receive a share of the farm's profits π (see below). Second-period labour endowments are $e_{h2} = 1$, and each household chooses food consumption c_{ht} , labour supply $\ell_{ht} \in [0, e_{ht}]$, which trade at market prices p_t and w_t in period $t \in \{1, 2\}$ respectively. Each household's utility is

$$u_1(c_{h1}, \ell_{h1}) + \beta u_2(c_{h2}, \ell_{h2}).$$

Household h 's utility maximisation problem is

$$\max_{(\ell_{ht} \in [0, e_{ht}], c_{ht} \in \mathbb{R}_+)^2_{t=1}} u_1(c_{h1}, \ell_{h1}) + \beta u_2(c_{h2}, \ell_{h2}) \quad (74)$$

$$\text{s.t. } p_1 c_{h1} + p_2 c_{h2} = w_1 \ell_{h1} + w_2 \ell_{h2} + \frac{\pi}{|H|}. \quad (75)$$

Farm. The farm uses labour L_t in period t to produce $Y_t = f(L_t)$ units of food. The farm's profit function is

$$\pi(p_1, p_2, w_1, w_2) = \max_{L_1, L_2} p_1 f(L_1) + p_2 f(L_2) - w_1 L_1 - w_2 L_2. \quad (76)$$

Equilibrium. An equilibrium consists of prices (p_1, p_2, w_1, w_2) and quantities

$$(\{c_{ht}, \ell_{ht}\}_{(h,t) \in H \times \{1,2\}}, L_1, L_2, Y_1, Y_2)$$

such that the quantities solve the respective problems above, and all four markets clear:

$$\sum_{h \in H} \ell_{h1} = L_1 \quad (77)$$

$$\sum_{h \in H} \ell_{h2} = L_2 \quad (78)$$

$$\sum_{h \in H} c_{h1} = Y_1 \quad (79)$$

$$\sum_{h \in H} c_{h2} = Y_2. \quad (80)$$

- (ii) Suppose that at the (non-equilibrium) market prices, the market values of the excess demands for food sum to a positive number. Prove that there is excess supply in at

least one of the labour markets. *Note: do not assume that there is excess demand in both food markets.*

Answer. One form of Walras' law is that for every vector of prices (p_1, p_2, w_1, w_2) ,

$$\sum_{h \in H} [p_1 c_{h1} + p_2 c_{h2} - w_1 \ell_{h1} - w_2 \ell_{h2}] - p_1 Y_1 - p_2 Y_2 + w_1 L_1 + w_2 L_2 = 0. \quad (81)$$

(This is obtained by summing all households' budget constraints and substituting in the farm's profits π .) Now, if the market value of the excess demand for food is

$$\sum_{h \in H} [p_1 c_{h1} + p_2 c_{h2}] - p_1 Y_1 - p_2 Y_2 \quad (82)$$

is positive, then the remaining terms (the market value of the excess demand for labour)

$$w_1 L_1 + w_2 L_2 - \sum_{h \in H} [w_1 \ell_{h1} + w_2 \ell_{h2}] \quad (83)$$

must be negative. This means there must be excess supply of labour in at least one time period.

- (iii) Prove that the farm reacts to second-period food-price increases by increasing supply.

Answer. By the envelope theorem,

$$\frac{\partial \pi(p_1, p_2, w_1, w_2)}{\partial p_2} \quad (84)$$

$$= \frac{\partial}{\partial p_2} [p_1 f(L_1) + p_2 f(L_2) - w_1 L_1 - w_2 L_2] \Big|_{L_1=L_1(p_1, p_2, w_1, w_2), L_2=L_2(p_1, p_2, w_1, w_2)} \quad (85)$$

$$= f(L_2) \Big|_{L_2=L_2(p_1, p_2, w_1, w_2)} \quad (86)$$

$$= Y_2(p_1, p_2, w_1, w_2). \quad (87)$$

Since π is the upper envelope of a set of linear functions of prices (one for each (L_1, L_2) choice), π is a convex function. Therefore, the left side is increasing in p_2 , so the right side is also increasing in p_2 . We conclude that the firm's second-period food supply is increasing in the second period food price.

- (iv) Write down the utility maximisation problem of a "big family" household that makes all market transactions on behalf of the households and the farm. Assume that the big-family household puts equal utility weight on all actual households. *Hints. Recall the home-production example from class. Put the market transactions in one Bellman equation, put the farm choices inside another Bellman equation, and bury the allocation of resources to households inside a value function.*

Answer. Upper case letters (L_1 , etc.) are quantities that are traded, hatted letters (\hat{L}_1 , etc.) are production quantities, and lower case letters (ℓ_{h1} , etc.) are household quantities. The big household's problem is:

$$\begin{aligned} & \max_{L_1, L_2, Y_1, Y_2} U(L_1, L_2, Y_1, Y_2) \\ & \text{s.t. } p_1 Y_1 + p_2 Y_2 = w_1 L_1 + w_2 L_2 \end{aligned}$$

where

$$\begin{aligned}
 U(L_1, L_2, Y_1, Y_2) &= \max_{\hat{L}_1, \hat{L}_2, \hat{Y}_1, \hat{Y}_2} V(\hat{L}_1, \hat{L}_2, \hat{Y}_1, \hat{Y}_2) \\
 \text{s.t.} \quad & f(\hat{L}_1 - L_1) + Y_1 = \hat{Y}_1 \\
 & f(\hat{L}_2 - L_2) + Y_2 = \hat{Y}_2
 \end{aligned}$$

and

$$\begin{aligned}
 V(\hat{L}_1, \hat{L}_2, \hat{Y}_1, \hat{Y}_2) &= \max_{\ell_{ht} \in [0, e_{ht}], c_{ht} \in \mathbb{R}_+} \sum_{h \in H} [u_1(c_{h1}, \ell_{h1}) + \beta u_2(c_{h2}, \ell_{h2})] \\
 \text{s.t.} \quad & \sum_{h \in H} \ell_{h1} = \hat{L}_1 \\
 & \sum_{h \in H} \ell_{h2} = \hat{L}_2 \\
 & \sum_{h \in H} c_{h1} = \hat{Y}_1 \\
 & \sum_{h \in H} c_{h2} = \hat{Y}_2.
 \end{aligned}$$

- (v) Suppose the government forcibly reallocated all resources to an efficient egalitarian allocation. If the population were allowed to trade based on their new endowments, what competitive allocation would arise?

Answer. No trade would be an equilibrium, i.e. the egalitarian allocation would become a competitive allocation. This is essentially the second welfare theorem, whose proof can be adapted to this situation as follows.

An equilibrium based on the new (egalitarian) endowments exists. (By the previous question, the economy can be reformulated as a pure-exchange equilibrium with a single household, and the pure-exchange existence theorem would apply.) Since the egalitarian allocation is the endowment, no household can be worse off under the new equilibrium allocation. Since the egalitarian allocation is efficient, this means that no household can be strictly better off. Therefore, every household is indifferent between the egalitarian allocation and the new equilibrium allocation. Thus, under the new equilibrium prices, the egalitarian allocation is an equilibrium allocation.

- (vi) * Give an example of a metric space with the property that every closed subset is compact.

Answer. Any compact metric space, for example $([0, 1], d_2)$. This is because in every metric space (X, d) , every closed subset A of a compact set K is compact. We proved this in class:

Let $a_n \in A$ be any sequence. Since $a_n \in K$ and K is compact, a_n has a convergent subsequence $b_n \rightarrow b$ with $b \in K$. Since A is closed, $b \in A$. We conclude that A is compact.

(vii) * Prove the Cantor intersection theorem:

Let (X, d) be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) A_n is compact for all n . Let $A = \bigcap A_n$. Then $A \neq \emptyset$.

Answer. See the proof of Cantor's intersection theorem in the notes.

26: Micro 1, May 2017

A café and a restaurant both serve meals to customers, using labour and food. The restaurant requires double the labour and food inputs to produce the same number of meals as the café. Households supply labour and only eat at restaurants and/or cafes. At every level of consumption and supply, households prefer an extra restaurant meal to an extra café meal. A farm produces food from labour only.

- (i) Write down a competitive equilibrium model of the labour, food, and meals (restaurants and cafes) markets.

Answer.

Households choose restaurant and café meals m^r and m^c and work hours h at prices p^r, p^c, w respectively to maximise utility $u(m^r, m^c, h)$. There are n households, and each household receives an equal share of all firms' profits Π . The representative household's utility maximisation problem is:

$$\max_{m^r, m^c, h} u(m^r, m^c, h) \quad (88)$$

$$\text{s.t. } p^r m^r + p^c m^c = wh + \Pi/n. \quad (89)$$

Firms. There are three firms, with total profits $\Pi = \pi^f + \pi^c + \pi^r$ arising from the farm, café, and restaurant, respectively. The **farm** uses H^f hours of labour to produce $Y = f(H^f)$ units of food which it sells at price p^y . Its profit function is

$$\pi^f(p^y, w) = \max_{H^f} p^y f(H^f) - wH^f. \quad (90)$$

The **café** uses H^c hours of labour and y^c units of food to produce $M^c = g(H^c, y^c)$ café meals. Its profit function is

$$\pi^c(p^c, p^y, w) = \max_{H^c, y^c} p^c g(H^c, y^c) - wH^c - p^y y^c. \quad (91)$$

The **restaurant** uses H^r hours of labour and y^r units of food to produce $M^r = g(H^r/2, y^r/2)$ restaurant meals. Its profit function is

$$\pi^r(p^r, p^y, w) = \max_{H^r, y^r} p^r g(H^r/2, y^r/2) - wH^r - p^y y^r. \quad (92)$$

An **equilibrium** consists of prices (p^y, p^c, p^r, w) and quantities

$$(m^r, m^c, h, H^f, H^c, H^r, M^c, M^r, Y, y^c, y^r)$$

such that the quantities solve the respective problems above given these prices, and all four markets clear:

$$y^c + y^r = Y \quad (93)$$

$$nm^c = M^c \quad (94)$$

$$nm^r = M^r \quad (95)$$

$$nh = H^c + H^r + H^f. \quad (96)$$

- (ii) Suppose there is an equilibrium in which restaurant meals cost £1. Does this mean that there is an equilibrium in which café meals cost £1?

Comment. Most students misinterpreted the question as asking: is there an equilibrium in which both types of meals cost £1? The first part of the question consists of almost irrelevant information. However, it is not completely irrelevant: it rules out the possibility that there is no equilibrium at all.

Answer. Yes. Let $p = (p^y, p^c, p^r, w)$ be the prices in the equilibrium for which restaurant meals cost $p^r = 1$. Then the price vector p/p^c combined with the same quantities give an equilibrium in which café meals cost 1.

- (iii) Prove that in every equilibrium in which café meals are sold, restaurant meals trade at a higher price than café meals.

Answer. If café meals were more or equally expensive, then households would not buy them.

- (iv) Prove that the marginal cost of restaurant meals equals the wage divided by the marginal productivity of labour.

Answer. There are at least two approaches: (1) construct the cost function of a lazy manager who demands the same amount of food regardless of the meal production target or price, or (2) apply the Lagrange-multiplier version of the envelope theorem to the cost function, and calculate the relevant multipliers using first-order conditions. All correct answers from students used the second approach. I pursue the first option.

The restaurant's profit function can be decomposed into input and output choices:

$$\pi^r(p^r, p^y, w) = \max_{M^r} p^r M^r - C^r(M^r; p^y, w) \quad (97)$$

where

$$C^r(M^r; p^y, w) = \min_{H^r, y^r} wH^r + p^y y^r \quad (98)$$

$$\text{s.t. } g(H^r/2, y^r/2) \geq M^r. \quad (99)$$

Suppose that $(\bar{M}^r; \bar{p}^y, \bar{w})$ are the equilibrium production target and prices, and that the best choices are (\bar{H}^r, \bar{y}^r) . Now, consider a lazy manager who only adjusts the labour demand in response to a production target or price change. His value function is:

$$L^r(M^r; p^y, w) = wH^r(M^r; \bar{M}^r, \bar{p}^y, \bar{w}) + p^y \bar{y}^r, \quad (100)$$

where $H^r(M^r; \bar{y}^r)$ is the lazy demand function defined implicitly the equation

$$g(H^r/2, \bar{y}^r/2) = M^r. \quad (101)$$

By the implicit function theorem, the lazy manager's marginal cost is

$$\frac{\partial L^r(M^r; p^y, w)}{\partial M^r} = w \frac{\partial H^r(M^r; \bar{M}^r, \bar{p}^y, \bar{w})}{\partial M^r} \quad (102)$$

$$= -w \frac{-1}{g_1(H^r(M^r; \bar{y}^r)/2, \bar{y}^r/2)} \quad (103)$$

$$= \frac{w}{g_1(H^r(M^r; \bar{y}^r)/2, \bar{y}^r/2)}. \quad (104)$$

Since the lazy value function is tangent to the cost function (e.g. due to the differentiable sandwich lemma), we have

$$\left. \frac{\partial C^r(M^r; p^y, w)}{\partial M^r} \right|_{(M^r; p^y, w) = (\bar{M}^r; \bar{p}^y, \bar{w})} \quad (105)$$

$$= \left. \frac{\partial L^r(M^r; p^y, w)}{\partial M^r} \right|_{(M^r; p^y, w) = (\bar{M}^r; \bar{p}^y, \bar{w})} \quad (106)$$

$$= \frac{\bar{w}}{g_1(\bar{H}^r/2, \bar{y}^r/2)}. \quad (107)$$

(v) Prove that the restaurant's marginal cost curve is increasing.

Answer. The marginal cost curve is given by

$$C^r(M^r; p^y, w) = \min_{H^r, y^r} wH^r + p^y y^r \quad (108)$$

$$\text{s.t. } g(H^r/2, y^r/2) \geq M^r. \quad (109)$$

We check that it is convex, based on the assumption that g is a concave production function. Let (H, y) and (H', y') be optimal plans to meet production targets M and M' respectively. Then

$$tC^r(M; p^y, w) + (1-t)C^r(M'; p^y, w) = t(wH + p^y y) + (1-t)(wH' + p^y y') \quad (110)$$

$$= w(tH + (1-t)H') + p^y(ty + (1-t)y) \quad (111)$$

$$\geq C^r(tM + (1-t)M'; p^y, w). \quad (112)$$

The last step is true because the intermediate plan $t(H, y) + (1-t)(H', y')$ meets the intermediate target $tM + (1-t)M'$, but is not necessarily the lowest cost plan to do so. This is because g is a concave production function.

(vi) The government would like to increase restaurant meal consumption. It proposes (symmetric) lump-sum transfer scheme from households to the restaurant. Would this policy have the desired effect?

Comment. Answers to this question were often misleading and/or incomplete. For example, one student wrote:¹

¹I simplified the English a little bit.

No, under the first welfare theorem, the equilibrium is already efficient and no lump-sum tax transfer under the same feasibility constraints can lead to a Pareto improvement.

These two statements are true, but incomplete. In particular, they do not rule out the possibility that policy might increase restaurant meal consumption either (i) by delivering an inefficient allocation, (ii) by making someone better off and someone else worse off, or (iii) by having no effect on welfare at all.

Answer. No, it would have no effect at all. Since the households hold equal shares in the restaurant, each household would have a net tax of zero. Adding a constant to the firm's objective does not affect its optimal choices.

- (vii) * Give an example of a metric space with the property that every closed subset is compact.

Answer. This is a repeat from the previous question.

- (viii) * Prove the Cantor intersection theorem:

Let (X, d) be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) A_n is compact for all n . Let $A = \bigcap A_n$. Then $A \neq \emptyset$.

Answer. See the proof of Cantor's intersection theorem in the notes.

27: AME, December 2017

Part A

Internet data centres generate waste energy that can be used to heat homes. Suppose that this waste energy can be transported but not stored. Households benefit more from heat during the evening, and benefit more from the Internet during the day. Households own the data centres, which they rent out to the internet company.

- (i) Write down a competitive equilibrium model of the data centre, computing and heat markets during the day and evening.

Comment. Most students answered this part well. Most of the mistakes were items from the checklist (see the start of this document) and did not relate specifically to the structure of this question.

Answer. Households. There are n identical households, and two time periods ($t = 1$ is day and $t = 2$ is night). Each household is endowed with equipment e which they rent at prices r_t , and buy heat h_t metered at price m_t , and internet services i_t at price p_t . These choices give them utility $u_1(h_1, i_1) + u_2(h_2, i_2)$. Each household receives a dividend π/n . The utility maximisation problem is

$$\begin{aligned} \max_{h_1, i_1, h_2, i_2} & u_1(h_1, i_1) + u_2(h_2, i_2) \\ \text{s.t.} & m_1 h_1 + m_2 h_2 + p_1 i_1 + p_2 i_2 = r_1 e + r_2 e + \pi/n. \end{aligned}$$

Internet firm. The internet firm buys equipment E_t and produces heat $H_t(E_t)$ and internet services $I_t(E_t)$ in time t . Its profit function is

$$\begin{aligned} & \pi(r_1, r_2, m_1, m_2, p_1, p_2) \\ & = \max_{E_1, E_2} m_1 H_1(E_1) + m_2 H_2(E_2) + p_1 I_1(E_1) + p_2 I_2(E_2) - r_1 E_1 - r_2 E_2. \end{aligned}$$

Equilibrium. An equilibrium consists of prices $(r_1, r_2, m_1, m_2, p_1, p_2)$ and quantities $(E_1, E_2, h_1, i_1, h_2, i_2)$ such that all choices solve the problems above, and all markets clear:

$$\begin{aligned} ne &= E_1 \\ ne &= E_2 \\ nh_1 &= H_1(E_1) \\ nh_2 &= H_2(E_2) \\ ni_1 &= I_1(E_1) \\ ni_2 &= I_2(E_2). \end{aligned}$$

- (ii) Prove that if the daytime heating price increases, then the firm sells more daytime internet services.

Comment. Most students answered this question well.

Answer. By the envelope theorem,

$$\begin{aligned} & \frac{\partial \pi(r_1, r_2, m_1, m_2, p_1, p_2)}{\partial m_1} \\ &= \left[\frac{\partial}{\partial m_1} (m_1 H_1(E_1) + m_2 H_1(E_2) + p_1 I_1(E_1) + p_2 I_2(E_2) - r_1 E_1 - r_2 E_2) \right]_{\text{at optimal } (E_1, E_2)} \\ &= [H_1(E_1)]_{\text{at optimal } E_1} \\ &= H_1(E_1(r_1, r_2, m_1, m_2, p_1, p_2)). \end{aligned}$$

Now, since the profit function π is the upper envelope of linear functions (one function for each input choice (E_1, E_2)), we conclude that π is a convex function. Therefore the left side is increasing in m_1 . So the right side is increasing in m_1 .

We conclude that the demand for equipment E_1 would increase, and hence internet services $I_1(E_1)$ would increase.

- (iii) Write down a Bellman equation that separates the household's problem into day and evening choices.

Comment. Most students struggled with this question. We didn't go into much detail for this type of dynamic programming in class this year, but there are many practice questions like this.

Answer. Let a be assets saved from the first period. Then the utility maximisation problem can be decomposed into:

$$\begin{aligned} v_1(r_1, r_2, m_1, m_2, p_1, p_2) &= \max_{h_1, i_1, a} u_1(h_1, i_1) + v_2(a; r_2, m_2, p_2) \\ &\text{s.t. } m_1 h_1 + p_1 i_1 + a = r_1 e + \pi/n, \end{aligned}$$

where

$$\begin{aligned} v_2(a; r_2, m_2, p_2) &= \max_{h_2, i_2} u_2(h_2, i_2) \\ &\text{s.t. } m_2 h_2 + p_2 i_2 = r_2 e + a. \end{aligned}$$

Part B

Comment. Overall, most students answered many questions, but answered few questions well. In other words, most students need to improve the way they write their proofs.

- (i) Let (X, d_X) and (Y, d_Y) be complete metric spaces. Let $Z = X \times Y$ and $d_Z((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$. Prove that (Z, d_Z) is a complete metric space.

Comment. Most students struggled with this question. While many students seemed to have good intuition about what this question was about, they could not translate this into a logical explanation. The most common mistake was to start by assuming that x_n and y_n are Cauchy sequences. The proof needs to start by assuming that z_n is a Cauchy sequence. This *implies* that x_n and y_n are Cauchy sequences (but this requires a proof).

Answer. Let $z_n \in Z$ be a Cauchy sequence. We need to prove that z_n is convergent. Since $z_n = (x_n, y_n)$ is a Cauchy sequence, it follows that x_n is a Cauchy sequence. Specifically, pick any $r > 0$. Then there exists some N such that:

- $d_Z(x_n, y_n; x_m, y_m) < r$ for all $n, m > N$, and hence
- $\max\{d_X(x_n, x_m), d_Y(y_n, y_m)\} < r$ for all $n, m > N$, and hence
- $d_X(x_n, x_m) < r$ for all $n, m > N$.

Since x_n is a Cauchy sequence inside the complete metric space (X, d_X) , we conclude that x_n is convergent, i.e. $x_n \rightarrow x^*$ for some $x^* \in X$. By similar reasoning, $y_n \rightarrow y^*$ for some $y^* \in Y$.

It remains to show that $(x_n, y_n) \rightarrow (x^*, y^*)$. Pick any $r > 0$. Since $x_n \rightarrow x^*$, there exists some N_x such that for all $n > N_x$, $d_X(x_n, x^*) < r$. Since $y_n \rightarrow y^*$, there exists some N_y such that for all $n > N_y$, $d_Y(y_n, y^*) < r$. Therefore, for all $n > N = \max\{N_x, N_y\}$,

$$d_Z(x_n, y_n; x^*, y^*) = \max\{d_X(x_n, x^*), d_Y(y_n, y^*)\} < r.$$

- (ii) Let (X, d) be any metric space, and $A \subseteq X$ any subset. Provide a counter-example to the following false statement: the interior of the boundary of A is empty, i.e. $\text{int}(\partial A) = \emptyset$.

Comment. No student answered this question correctly. To make the question a bit easier, I could have added a hint: “think about incomplete metric spaces”.

Answer. Consider the metric space (\mathbb{R}, d_2) and the subset $A = \mathbb{Q}$. Then $\partial A = \mathbb{R}$, and $\text{int}(\partial A) = \mathbb{R}$.

We verify that $\partial A = \mathbb{R}$. First, if $x \in \mathbb{R} \setminus A$ (i.e. x is irrational), then

- there is a sequence $a_n \in A$ with $a_n \rightarrow x$ (based on the decimal expansion of x , and
- the trivial sequence $b_n = x \in \mathbb{R} \setminus A$ has $b_n \rightarrow x$.

So in this case, $x \in \partial A$.

Second, if $x \in A$ (i.e. x is rational), then

- the trivial sequence $a_n = x \in A$ has $a_n \rightarrow x$, and
- the sequence $b_n = x + \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus A$ and $b_n \rightarrow x$.

- (iii) Prove that if x is a boundary point of A in (X, d) (defined in terms of sequences), then every open neighbourhood U of x has $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$.

Comment. Most students did not know what “open neighbourhood” means, incorrectly thinking it meant an open ball containing x . But this did not matter much, because any open neighbourhood of x would contain an open ball of radius r centred at x .

Most students failed to connect the radius of the open ball, r , with the definition of convergence of sequences.

Answer. Suppose $x \in \partial A$, and pick any open neighbourhood U of x .

Since $x \in \partial A$, there exists sequences $a_n \in A$ and $b_n \notin A$ such that $a_n \rightarrow x$ and $b_n \rightarrow x$. Since x is in the interior of U , there is an open ball $N_r(x) \subseteq U$.

Now, because $a_n \rightarrow x$, it follows that there exists some N_a such that for all $n \geq N_a$, $d(a_n, x) < r$. Similarly, there is some N_b such that for all $n \geq N_b$, $d(b_n, x) < r$.

We conclude that $a_{N_a} \in U \cap A$ and $b_{N_b} \in U \cap (X \setminus A)$.

- (iv) Prove that $f : X \rightarrow Y$ is continuous if and only if for every open ball $U = N_r(y)$, the inverse image $f^{-1}(U)$ is an open set.

Comment. Most students adapted the open ball characterisation of continuity from class. This is fine, but I think it's easier to work with the open set characterisation of continuity.

Answer. We will make use of the following theorem: f is continuous if and only if for every open set $U \subseteq Y$, $f^{-1}(U)$ is an open set.

Since f is continuous, and pick any open ball $U = N_r(y)$. Since U is an open set, the theorem implies that $f^{-1}(U)$ is an open set.

Now suppose that $f^{-1}(U)$ is open for every open ball $U = N_r(y)$. We must prove that f is continuous. Let $V \subseteq Y$ be any open set. By the theorem, it suffices to show that $f^{-1}(V)$ is an open set. Since V is an open set, every point $v \in V$ is an interior point. Therefore, there exists an open ball $B_v = N_{r_v}(v) \subseteq V$ for every $v \in V$. Notice that $V = \cup_{v \in V} B_v$. Therefore

$$f^{-1}(V) = \cup_{v \in V} f^{-1}(B_v).$$

By the condition, each $f^{-1}(B_v)$ is an open set. Moreover, the union of open sets is open. We conclude that $f^{-1}(V)$ is an open set.

- (v) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$, where distances in both the domain and co-domain are measured with the Euclidean metric. Suppose that $\lim_{n \rightarrow \infty} f(1/n) = 1$ and $f(0) = 0$. Provide an example of an open set U such that $f^{-1}(U)$ is not open.

Comment. No student answered this question correctly.

Answer. Let $U = N_{0.5}(1) = (-\frac{1}{2}, \frac{1}{2})$ and $V = f^{-1}(U)$. Since U is an open ball, it is an open set. We will show that V is not an open set, contradicting the open set characterisation of continuity.

Observe that $0 \in V$, since $f(0) = 0$. We will find a sequence $z_n \notin V$ such that $z_n \rightarrow 0$, and we will conclude that $0 \in \partial V$ and hence that V is not an open set.

Let $x_n = 1/n$ and $y_n = f(1/n)$. Since $y_n \rightarrow 1$, there is a number N such that $d_2(y_n, 1) < 0.5$ for all $n > N$. So $y_{n+N} \notin U$, and hence $x_{n+N} \notin V$. So let $z_n = x_{n+N}$. Since $z_n \notin V$ is a subsequence of x_n , it follows that $z_n \rightarrow 0$.

- (vi) Let x_t be the fraction of women that work in professional occupations. Assume that this changes over time according to $x_{t+1} = f(x_t)$ where $f : [0, 1] \rightarrow [0, 1]$, and distances are measured by the Euclidean metric. Now, suppose that (i) f is

continuous, and that (ii) f is a contraction on $[0, \frac{1}{3})$, and also on $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, 1]$. Prove that there are either two or three steady-states (i.e. fixed points of f).

Comment. The most common mistakes in this question were:

- Applying Banach's fixed point theorem on $[0, \frac{1}{3})$, which is impossible since this is an incomplete metric space.
- Not accounting for the possibility that fixed points could lie at $\frac{1}{3}$ or $\frac{2}{3}$.

Answer. Suppose f is a contraction of degree a on these three sets.

First, we claim that f is a contraction on $[0, \frac{1}{3}]$. Recall from a homework problem that every distance metric is continuous, i.e. if $x_n \rightarrow x^*$ then $d(x_n, x_0) \rightarrow d(x^*, x_0)$. Let $x_n = \frac{1}{3} - \frac{1}{n}$. Since f is a contraction on $[0, \frac{1}{3})$, we know that for all $y \in [0, \frac{1}{3})$,

$$d(f(x_n), f(y)) \leq ad(x_n, y),$$

or equivalently, $g(x_n) \leq 0$ where

$$g(x) = d(f(x), f(y)) - ad(x, y).$$

Now, since g is continuous, it follows that $g(\frac{1}{3}) \leq 0$, and hence

$$d(f(\frac{1}{3}), f(y)) \leq ad(\frac{1}{3}, y).$$

So f is a contraction on $[0, \frac{1}{3}]$.

Similar logic establishes that f is a contraction on $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. These are closed subsets of \mathbb{R} , so they form complete metric spaces under the Euclidean metric.

By Banach's fixed point theorem there is exactly one fixed point each on the ranges $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. So, there are either three fixed points, or two with one of the fixed points being "shared" between two of these intervals, e.g. with $f(\frac{1}{3}) = \frac{1}{3}$.

- (vii) Investors with £200000 of assets are able to acquire visas (under some other conditions) to migrate to the United Kingdom. People residing outside the UK receive labour income w each period, and choose how much to consume c and save a' , and whether to migrate to the UK. Their utility each period is $u(c)$, which is discounted at rate β . Let $M(a)$ be the value of living in the UK as a migrant with assets a . Both u and M are bounded and concave. The value of assets to a foreigner $V(a)$ is characterised by the Bellman equation

$$V(a) = \begin{cases} M(a) & \text{if } a \geq 200000 \\ \max_{a'} u(a + w - a') + \beta V(a') & \text{if } a \in [0, 200000). \end{cases}$$

You hope to prove that V is concave with the following strategy – which turns out not to work:

- Prove that the Bellman operator is a contraction in $(B(\mathbb{R}_+), d_\infty)$.
- Prove that (X, d_∞) is a complete metric space, where

$$X = \{V \in B(\mathbb{R}_+) : V \text{ is concave and } V(a) = M(a) \text{ for } a \geq 200000\}.$$

- (c) Prove that the Bellman operator is a self-map on X .
- (d) Apply Banach's fixed point theorem.

Which step(s) succeed and which step(s) fail? *You will get credit for checking as many steps as you can.*

Comment. For the first part, most students completely ignored $M(a)$, and just replicated the proof of Blackwell's lemma.

For the second part, a common mistake was to think about the convergence of asset holdings of time. This is an interesting thing to study, but is not about the question at all. Convergence of value functions is about repeatedly refining an initial guess. Few people attempted the third part.

Answer.

- (a) This step fails, although it can be fixed with minor amendments.
Let $\Gamma(V)$ be the Bellman operator. If the value function V is poorly behaved (by being discontinuous), there might not be an optimal choice at a , and hence $\Gamma(V)(a)$ is not well-defined.
On the other hand, if we amend the Bellman equation by replacing "max" with "sup", then Blackwell's Lemma can be adapted to make this step work.
- (b) This step works. It suffices to prove that X is a closed set. Suppose $V_n \in X$ and $V_n \rightarrow V^*$. We must show that $V^* \in X$, i.e. that V^* is concave and $V(a) = M(a)$ for $a \geq 200000$. Consider $a_0, a_1 \geq 0$ and $t \in [0, 1]$. Since each V_n is concave, we know that

$$tV_n(a_0) + (1 - t)V_n(a_1) \leq V_n(ta_0 + (1 - t)a_1).$$

Taking limits, we find that

$$tV^*(a_0) + (1 - t)V^*(a_1) \leq V^*(ta_0 + (1 - t)a_1).$$

Next, pick any $a \geq 200000$. Since each $V_n \in V$, we know that $V_n(a) = M(a)$. It follows that $V^*(a) = \lim_{n \rightarrow \infty} V_n(a) = M(a)$.

- (c) This step fails. In particular $\Gamma(M)$ is discontinuous at 200000, and hence not concave. So $M \in X$ but $\Gamma(M) \notin X$, so Γ is not a self-map on X .
 - (d) This step would succeed if the other steps all worked. Specifically, suppose that (i) X were a closed subset of a complete metric space (and hence complete itself), and (ii) $\Gamma : X \rightarrow X$ were a self-map. These imply that Γ is a contraction on X , and by Banach's fixed point theorem would have a unique fixed point $V^* \in X$. This is the same fixed point as before, but we would then know that V^* is concave.
- (viii) Prove that the following optimization problem (relating to moral hazard) has an optimal solution:

$$\begin{aligned} & \min_{a, b \in \mathbb{R}_+} p \exp(a) + (1 - p) \exp(b) \\ & \text{s.t. } pa + (1 - p)b \geq 1, \text{ and} \\ & a - b \geq q, \end{aligned}$$

where $p \in (0, 1)$ and $q > 0$.

Comment. Nobody answered this question correctly.

One key trick that everybody missed is to think of the domain of (a, b) not as \mathbb{R}_+^2 , but rather as the set of (a, b) that satisfy the two constraints.

Another trick is to notice that the objective favours small (a, b) , but both must be positive. The goal is to remove some unfavourable items from the menu, and what's left over is hopefully compact. If you can remove big (a, b) from the menu, then that will help ensure the remaining menu is bounded. The hard bit is to make sure you are only remove suboptimal choices from the menu.

That's why I proposed removing items that are inferior to a particular choice.

Answer. Let C be the set of points (a, b) satisfying the first constraint, and D the points satisfying the second constraint. Since $C = f^{-1}([1, \infty))$ is the inverse image of a closed set on a continuous function $f(a, b) = pa + (1 - p)b$, we conclude that C is closed. Similarly, D is closed. So the set of feasible points, $C \cap D$ is closed.

Note that $(a, b) = (1 + q, 1) \in C \cap D$. So we can add a slack constraint that the point must be better than $(1 + q, 1)$, i.e.

$$p \exp(a) + (1 - p) \exp(b) \leq p \exp(1 + q) + (1 - p) \exp(1).$$

This new constraint implies

$$p \exp(a) + (1 - p) \exp(b) \leq \exp(1 + q) + \exp(1)$$

and hence

$$a \leq \log \frac{\exp(1 + q) + \exp(1)}{p}$$

and

$$b \leq \log \frac{\exp(1 + q) + \exp(1)}{1 - p}.$$

In other words, this new constraint ensures that the feasible points set is bounded. Then by the Bolzano-Weierstrass theorem, the feasible point set is compact.

The objective is continuous, so by the Extreme Value Theorem, a solution exists.

28: Micro 1, December 2017

Several identical households enjoy ice cream more in summer than winter, and enjoy soup in winter more than summer. Households are endowed with cows and fishing boats. Ice cream is made from cows. Soup is made from fishing boats. There are only two time periods (winter and summer).

- (i) Formulate a competitive equilibrium model of cows, boats, ice cream and soup during summer and winter.

Comment. A common mistake was to assume that households have the same preferences in summer and winter ($u_0 = u_1$ in my notation), even though the question explicitly said otherwise.

My sample solution does not specify exactly what u_0 and u_1 are. What matters is that this formulation is general enough to accommodate having different preferences in summer and winter. Note that a utility function of the form $u(i_0, s_0) + \beta u(i_1, s_1)$ is not enough, because it cannot express the idea that icecream is better in summer and soup is better in winter.

A common choice was to formulate the model using home production. This is fine, but it complicates everything. It's important to pick the simplest version of the model that you can.

Answer. Households. There are n identical households which live in periods $t \in \{0, 1\}$, where 0 is summer and 1 is winter. Each household is endowed with b boats and c cows, which it rents at prices r_t^b and r_t^c . It also receives dividends Π/n , where $\Pi = \pi^i + \pi^k$ (defined below). It spends these resources on icecream i_t and soup s_t at prices p_t^i and p_t^s , which give utility $u_0(i_0, s_0) + u_1(i_1, s_1)$. The households' maximisation problem is

$$\begin{aligned} & \max_{i_0, i_1, s_0, s_1} u_0(i_0, s_0) + u_1(i_1, s_1) \\ \text{s.t. } & i_0 p_0^i + i_1 p_1^i + s_0 p_0^s + s_1 p_1^s \leq (r_0^b + r_1^b)b + (r_0^c + r_1^c)c + \frac{\Pi}{n}. \end{aligned}$$

Dairy. A dairy rents C_t cows to produce $I_t = f(C_t)$ units of icecream in time period t . Its profit function is

$$\pi^i(p_0^i, p_1^i, r_0^c, r_1^c) = \max_{C_0, C_1} p_0^i f(C_0) + p_1^i f(C_1) - r_0^c C_0 - r_1^c C_1.$$

Kitchen. A kitchen rents B_t boats to produce $S_t = g(B_t)$ units of soup in time period t . Its profit function is

$$\pi^k(p_0^s, p_1^s, r_0^b, r_1^b) = \max_{B_0, B_1} p_0^s g(B_0) + p_1^s g(B_1) - r_0^b B_0 - r_1^b B_1.$$

Equilibrium. Prices $(p_t^i, p_t^s, r_t^c, r_t^b)_{t \in \{0,1\}}$ and quantities $(i_t, s_t, B_t, C_t)_{t \in \{0,1\}}$ consti-

tute an equilibrium if

$$\begin{aligned}
 ni_0 &= f(C_0) \\
 ni_1 &= f(C_1) \\
 ns_0 &= g(B_0) \\
 ns_1 &= g(B_1) \\
 nb &= B_0 \\
 nb &= B_1 \\
 nc &= C_0 \\
 nc &= C_1.
 \end{aligned}$$

- (ii) Suppose that the boat, cow, and icecream markets clear. Does this imply that the soup markets clear?

Answer. No.

Comment. For example, there could be excess demand for soup in winter and excess supply in summer.

- (iii) Reformulate the households' problem by constructing a value function for both time periods, which are connected via a Bellman equation.

Answer. Consider the value of holding money m in period 1,

$$\begin{aligned}
 V_1(m_1, p_1^i, p_1^s, r_1^c, r_1^b) &= \max_{i_1, s_1} u_1(i_1, s_1) \\
 \text{s.t. } i_1 p_1^i + s_1 p_1^s &\leq r_1^b b + r_1^c c + m_1.
 \end{aligned}$$

Then the indirect utility function in period 0 is

$$\begin{aligned}
 V_0(p_0^i, p_0^s, r_0^c, r_0^b; p_1^i, p_1^s, r_1^c, r_1^b) &= \max_{i_0, s_0, m_1} u_0(i_0, s_0) + V_1(m_1, p_1^i, p_1^s, r_1^c, r_1^b) \\
 \text{s.t. } i_0 p_0^i + s_0 p_0^s + m_1 &\leq r_0^b b + r_0^c c + \frac{\Pi}{n}.
 \end{aligned}$$

- (iv) How does the winter supply of icecream change when the winter price of soup increases?

Comment. Most students did not read the question properly, and answered as if the question were asking about the supply of soup.

Answer. The winter price of soup does not appear in the dairy's problem, so it has no effect.

- (v) Is the Pareto frontier of this economy a convex set (under appropriate convexity assumptions about preferences and production)?

Comment. Most students got this wrong. Perhaps students were confusing the utility possibility set (which is convex, at least with free disposal) with its frontier.

Answer. No, assuming that the flow utility function u is strictly concave. Let \mathcal{U} be the Pareto frontier. Let $\hat{u} = u_0(f(C_0), g(B_0)) + u_1(f(C_1), g(B_1))$, and $\bar{u} =$

$u_0(0, 0) + u_1(0, 0)$. Allocating all resources to household n would give a utility vector of $U_n = (\hat{u}, \bar{u}, \dots, \bar{u})$. Since these are efficient, we know that each $U_n \in \mathcal{U}$.

Now, consider $U' = \frac{1}{2}U_1 + \frac{1}{2}U_2$. In this case, households 1 and 2 would receive utility

$$\begin{aligned} & \frac{1}{2}[u_0(0, 0) + u_1(0, 0)] + \frac{1}{2}[u_0(f(C_0), g(B_0)) + u_1(f(C_1), g(B_1))] \\ & < u_0(\frac{1}{2}f(C_0), \frac{1}{2}g(B_0)) + u_1(\frac{1}{2}f(C_1), \frac{1}{2}g(B_1)) \\ & = u^*. \end{aligned}$$

So households 1 and 2 strictly prefer u^* over $\frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}$. This means that $(u^*, u^*, \bar{u}, \dots, \bar{u})$ is feasible and Pareto dominates U' . So $U' \notin \mathcal{U}$. We conclude that \mathcal{U} is not a convex set.

- (vi) Prove that there is only one competitive equilibrium (under appropriate convexity assumptions about preferences and production).

Comment. There are two critical ingredients: (i) equilibria are efficient (the first welfare theorem) and (ii) equilibria are symmetric if households are all the same. Most students missed one of these ingredients out.

Answer. Assume the utility function is strictly concave and the production is strictly increasing.

By the first welfare theorem, all equilibrium are Pareto efficient. Since households are identical, they all acquire the same utility in equilibrium. Since the equilibrium allocation is efficient and gives equal utility to all households, it be a solution to the egalitarian social planner's problem,

$$\begin{aligned} & \max_{(i_{ht}, s_{ht})_{h \in \{1, \dots, n\}, t \in \{0, 1\}}} \sum_{h=1}^n [u_0(i_{h0}, s_{h0}) + u_1(i_{h1}, s_{h1})] \\ \text{s.t. } & \sum_{h=1}^n i_{h0} = f(nc) \\ & \sum_{h=1}^n i_{h1} = f(nc) \\ & \sum_{h=1}^n s_{h0} = g(nb) \\ & \sum_{h=1}^n s_{h1} = g(nb). \end{aligned}$$

Since the constraint is linear and the objective is strictly concave, it has a unique solution.

- (vii) * Consider any metric space (X, d) , and any two sets A and B with $A \subseteq B \subseteq X$. Prove that if A is open in (X, d) , then A is open in (B, d) .

Comment. Most students did not attempt this question.

Answer. Pick any $a \in A$. We must find some $r > 0$ such that

$$\{b \in B : d(a, b) < r\} \subseteq A.$$

Since A is open in (X, d) , there is some $s > 0$ such that

$$\{x \in X : d(a, x) < s\} \subseteq A.$$

Pick $r = s$. Since $B \subseteq X$, it follows that

$$\{x \in B : d(a, x) < r\} \subseteq \{x \in X : d(a, x) < r\} \subseteq A$$

as required.

- (viii) * Let $f : X \rightarrow X$ be a function on a complete metric space (X, d) . Suppose that $g(x) = f(f(x))$ is a contraction. Prove that f has a unique fixed point.

Comment. Most students did not attempt this question.

Answer. By Banach's fixed point theorem, g has a unique fixed point, x^* . So $f(f(x^*)) = x^*$. Since fixed points of f are also fixed points of g , we conclude that x^* is the only possible fixed point of f .

It remains to show that x^* is a fixed point of f . Now, consider $y = f(x^*)$. We need to prove that $y = x^*$. Notice that

$$g(y) = g(f(x^*)) = f(f(f(x^*))) = f(g(x^*)) = f(x^*) = y.$$

So y is also a fixed point of g . But g has exactly one fixed point, namely x^* . So we conclude that $y = x^*$, as required.

29: AME, May 2018

Part A

Are improvements in renewable energy technology good news for climate change?

Suppose households own oil deposits, which they can extract and sell at any time. In the first year, only the oil-based power firm operates; it buys oil from households. In the second year, the solar-based power firm is able to hire workers to run solar plants. A pharmaceutical firm hires workers and uses electricity to make medicine. Households only consume electricity and medicine.

- (i) Write down a competitive equilibrium model of the power and pharmaceutical industries.

Comment. Most students' answers were overly complicated. For example, many students put leisure into the utility function, even though the question said that households only consume electricity and medicine. This did not lead to losing marks, although I suspect many such students confused themselves and made important mistakes because of the over-complication.

The most common mistake was to neglect the assumption that the household can choose when to extract his oil deposits, i.e. oil is storable.

Answer. There are two time periods, $t \in \{1, 2\}$ and n identical **households**. At time t , each household sells hours of labour h_t at price w_t , barrels of oil b_t at price r_t and buys electricity z_t at price q_t and medicine m_t at price p_t . This leads to a utility at time t is $u_t(z_t, m_t)$. The household's oil endowment is e_t , and dividends from firm profits are $\frac{\Pi}{n} = \frac{\pi^o + \pi^s + \pi^m}{n}$. The *household's problem* is

$$\begin{aligned} \max_{h_t \in [0,1], b_t, z_t, m_t} \quad & u_1(z_1, m_1) + u_2(z_2, m_2) \\ \text{s.t.} \quad & \sum_{t=1}^2 q_t z_t + p_t m_t = \frac{\Pi}{n} + \sum_{t=1}^2 w_t h_t + r_t b_t \text{ and } b_1 + b_2 \leq e. \end{aligned}$$

In time t , the **oil-based power firm** purchases B_t^o barrels of oil, to produce $Z_t^o = f^o(B_t^o)$ units of electricity. The firm's profit function is

$$\pi^o(q_1, q_2; r_1, r_2) = \max_{B_t^o} \sum_{t=1}^2 q_t f^o(B_t^o) - r_t B_t^o$$

In time 2, the **solar-based power firm** hires H_2^s workers to produce $Z_2^s = f^s(H_2^s)$ units of electricity. The firm's profit function is

$$\pi^s(q_2; w_2) = \max_{H_2^s} q_2 f^s(H_2^s) - w_2 H_2^s$$

In time t , the **medicine firm** hires H_t^m workers and buys Z_t^m units of electricity to produce $M_t^m = f^m(H_t^m, Z_t^m)$ bottles of medicine. The firm's profit function is

$$\pi^m(p_1, p_2; q_1, q_2, w_1, w_2) = \max_{H_t^m, Z_t^m} \sum_{t=1}^2 p_t f^m(H_t^m, Z_t^m) - w_t H_t^m - q_t Z_t^m$$

An **equilibrium** consists of prices $(p_t^*, q_t^*, r_t^*, w_t^*)_{t=1}^2$ and quantities

$$(h_t^*, b_t^*, z_t^*, m_t^*, B_t^{o*}, H_2^{s*}, H_t^{m*}, Z_t^{o*}, Z_t^{s*}, Z_t^{m*}, M_t^{m*})_{t=1}^2$$

such that all markets clear:

$$\begin{aligned} nh_1^* &= H_1^{m*} \\ nh_2^* &= H_2^{s*} + H_2^{m*} \\ nb_1^* &= B_1^{o*} \\ nb_2^* &= B_2^{o*} \\ nz_1^* + Z_1^{m*} &= Z_1^{o*} \\ nz_2^* + Z_2^{m*} &= Z_2^{o*} + Z_2^{s*} \\ nm_1^* &= M_1^{m*} \\ nm_2^* &= M_2^{m*}. \end{aligned}$$

- (ii) Prove that the oil-based firm reacts to an energy price decrease in the second year (keeping all other prices fixed) by buying less oil in the second year.

Comment. This question was generally answered well.

Answer. By the envelope theorem,

$$\begin{aligned} & \frac{\partial}{\partial q_2} \pi^o(q_1, q_2; r_1, r_2) \\ &= \left[\frac{\partial}{\partial q_2} \left(\sum_{t=1}^2 q_t f^o(B_t^o) - r_t B_t^o \right) \right]_{\text{at optimal choices}} \\ &= [f^o(B_t^o)]_{B_2^o = B_2^o(q_1, q_2; r_1, r_2)} \\ &= Z_2^o(q_1, q_2; r_1, r_2). \end{aligned}$$

Now, π^o is the upper envelope of a set of linear functions – one for each choice of (B_1^o, B_2^o) . Therefore π^o is a convex function. This implies that the left side of the equation above (a derivative of π^o) is increasing in the electricity price q_2 . So the right side is also increasing in q_2 . Therefore, electricity supplied by the oil firm is increasing in the electricity price q_2 . Since the production function is increasing, we conclude that if electricity prices decrease, the firm's demand for oil decreases.

- (iii) How does the nature of the solar-based power firm's production function affect the equilibrium amount of oil extracted?

Comment. Most students focused on how the oil extraction occurred over the two time periods, rather than the total quantity extracted. I think the wording of the question is unambiguous. But more importantly, the whole exam question was explicitly motivated by the environmental impact of renewable energy research. The total amount of oil extracted is clearly more important than the timing (at least over the time horizon that research needs to happen at).

Answer. It has no effect. Since the price of oil must be strictly positive in the first period, households sell all oil (either in the first or second period), so all oil is extracted. This is bad news for global warming!

- (iv) Decompose the pharmaceutical firm's choices into input and output choices using a Bellman equation.

Comment. Many students overcomplicated their answers, by having a single cost function with two production targets.

Answer. The pharmaceutical firm's profit function can be written as

$$\pi^m(p_1, p_2; q_1, q_2, w_1, w_2) = \max_{M_t^m} \sum_{t=1}^2 p_t M_t^m - C^m(M_t^m; q_t, w_t)$$

where

$$C^m(M^m; q, w) = \min_{H^m, Z^m} wH^m + qZ^m$$

s.t. $f^m(H^m, Z^m) \geq M^m$.

Part B Comment. As usual, many students attempted many parts poorly, rather than trying to do a small number of parts well. It is possible to get a mark in the 80s by only answering two or three parts very well. (A mark in the 90s requires more breadth.) The reason for this is that these questions require some creativity (which is unpredictable under exam conditions), not just technical competence.

- (i) What is the interior of $A = \mathbb{Q}^2$ inside the metric space $(\mathbb{R} \times \mathbb{Q}, d_2)$? Recall that \mathbb{Q} is the set of rational numbers, and the Euclidean metric on this space is $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Comment. Almost no students got this question right, despite it being an easy question. I suspect most students would have had more success answering the almost identical question: what is the interior of \mathbb{Q} inside (\mathbb{R}, d_2) ? In other words, if an exam question seems hard, I recommend trying to answer an easier question first.

Answer. The interior of A is the empty set. Specifically consider any point $(x, y) \in A$. Every open ball $N_r(x, y) \subseteq \mathbb{R} \times \mathbb{Q}$ of radius $r > 0$ contains some point (x', y) such that $x' \notin \mathbb{Q}$ and hence $(x', y) \notin A$. So (x, y) is not an interior point. We conclude that A has no interior points.

- (ii) Consider the metric space $(\ell_\infty(\mathbb{R}), d_\infty)$, i.e. the set of sequences whose absolute values sum to a finite number. Provide an example of a contraction $f : \ell_\infty(\mathbb{R}) \rightarrow \ell_\infty(\mathbb{R})$. Recall that $d_\infty(\{x_n\}, \{y_n\}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$.

I give two possible answers. Of course, there are many more.

Answer 1. Let $z_n = 0$ be the trivial sequence consisting of zeros. The function $f(\{x_n\}) = \{z_n\}$ is a contraction of degree 0. In particular, $d_\infty(f(\{x_n\}), f(\{y_n\})) = 0 \leq d_\infty(\{x_n\}, \{y_n\})$.

Answer 2. Consider the function $f(\{x_n\}) = \frac{1}{2} \{x_n\}$. Then

$$\begin{aligned} d_\infty(f(\{x_n\}), f(\{y_n\})) &= d_\infty(\tfrac{1}{2} \{x_n\}, \tfrac{1}{2} \{y_n\}) \\ &= \sup_n |\tfrac{1}{2}x_n - \tfrac{1}{2}y_n| \\ &= \tfrac{1}{2} \sup_n |x_n - y_n| \\ &= \tfrac{1}{2} d_\infty(\{x_n\}, \{y_n\}). \end{aligned}$$

So f is a contraction of degree $\frac{1}{2}$.

(iii) Consider any metric space (X, d) and any function $a : X \rightarrow \mathbb{R}_{++}$. Let $F_a(X) = \{f : X \rightarrow \mathbb{R}, \sup_{x \in X} a(x)f(x) < \infty\}$ and

$$d_a(f, g) = \sup_{x \in X} a(x)d_2(f(x), g(x)).$$

Prove that $(F_a(X), d_a)$ is a metric space. (Boyd, 1990, Journal of Economic Theory used this space to study unbounded value functions.)

Comment. One common mistake was to only prove half of the first item, but not the converse. (It's an "if and only if".)

Most mistakes were in establishing the triangle inequality.

Answer.

(a) $d_a(f, g) = 0 \iff f = g$:

$$\begin{aligned} f = g &\iff f(x) = g(x) \text{ for all } x \in X \\ &\iff d_2(f(x), g(x)) = 0 \text{ for all } x \in X \\ &\iff a(x)d_2(f(x), g(x)) = 0 \text{ for all } x \in X \\ &\iff d_a(f, g). \end{aligned}$$

(b) $d_a(f, g) = d_a(g, f)$:

$$\begin{aligned} d_a(f, g) &= \sup_{x \in X} a(x)d_2(f(x), g(x)) \\ &= \sup_{x \in X} a(x)d_2(g(x), f(x)) \\ &= d_a(g, f). \end{aligned}$$

(c) $d_a(f, h) \leq d_a(f, g) + d_a(g, h)$:

$$\begin{aligned} d_a(f, h) &= \sup_{x \in X} a(x)d_2(f(x), h(x)) \\ &\leq \sup_{x \in X} a(x)[d_2(f(x), g(x)) + d_2(g(x), h(x))] \\ &= \sup_{x \in X} [a(x)d_2(f(x), g(x)) + a(x)d_2(g(x), h(x))] \\ &\leq \sup_{x \in X} a(x)d_2(f(x), g(x)) + \sup_{x \in X} a(x)d_2(g(x), h(x)) \\ &= d_a(f, g) + d_a(g, h). \end{aligned}$$

- (iv) Prove that the metric space $(F_a(X), d_a)$ as defined in the previous question is complete.

Comment. Most students confused the spaces, e.g. by choosing sequences inside X rather than $F_a(X)$.

Answer. Let $f_n \in F_a(X)$ be a Cauchy sequence. We need to prove that there is some $f^* \in F_a(X)$ such that $f_n \rightarrow f^*$.

Fix any $x \in X$. Observe that $x_n = f_n(x)$ is a Cauchy sequence inside (\mathbb{R}, d_2) . Since (\mathbb{R}, d_2) is complete, $f_n(x)$ converges to some point, which we will call $f^*(x)$. This means we have defined f^* for every point $x \in X$.

Next we show that $d_a(f_n, f^*) \rightarrow 0$. Since f_n is a Cauchy sequence, for every $r > 0$ there exists some N such that:

- for all $n, m > N$, $d_a(f_n, f_m) < r$,
- which means for all $n, m > N$, $\sup_{x \in X} a(x)d_2(f_n(x), f_m(x)) < r$,
- which implies, by continuity of d_2 , that for all $n > N$, $\sup_{x \in X} a(x)d_2(f_n(x), f^*(x)) < r$,
- and therefore for all $n > N$, $d_a(f_n, f^*) < r$.

Before we can conclude that $f_n \rightarrow f^*$, we need to verify that $f^* \in F_a(X)$, i.e. that $\sup_{x \in X} a(x)f^*(x) < \infty$. By the previous paragraph, for $r = 1$ there exists some N such that $d_a(f_N, f^*) < 1$. By the triangle inequality

$$d_a(0, f^*) \leq d_a(0, f_N) + d_a(f_N, f^*) \leq d_a(0, f_N) + 1.$$

Since $f_N \in F_a(X)$, the $d_a(0, f_N)$ term is finite. So $d_a(0, f^*)$ is finite, and $f^* \in F_a(X)$. We conclude that $f_n \rightarrow f^*$.

- (v) Suppose that a person of height $h \in [0, 1]$ has a utility function for food consumption $c \in [0, 2]$ of $u_h(c) = c^h$. Prove that the set of these utility functions, $U = \{u_h : h \in [0, 1]\}$ is a compact subset of the metric space $(CB([0, 2]), d_\infty)$.

Note: you can assume that $f(x, y) = x^y$ and similar functions are continuous. Recall that $CB([0, 2]) = \{f : [0, 2] \rightarrow \mathbb{R}, f \text{ is continuous and bounded}\}$, and $d_\infty(f, g) = \sup_{x \in \mathbb{R}_+} |f(x) - g(x)|$.

Comment. The key to answering this question is to think about the function that maps heights to utility functions. Most students missed this observation.

Answer. Consider the function $T : [0, 1] \rightarrow CB([0, 2])$ defined by $T(h)(c) = c^h$. I will prove that T is continuous (using the Euclidean metric on the domain), that $[0, 1]$ is compact, and conclude that the range, U is compact.

To show that T is continuous, pick any convergent sequence $h_t \in [0, 1]$, where $h_t \rightarrow h^*$. Let $f_t = T(h_t)$ and $f^* = T(h^*)$. We need to prove that $f_t \rightarrow f^*$. Now,

$$\begin{aligned} d_\infty(f_t, f^*) &= \sup_{c \in [0, 2]} |f_t(c) - f^*(c)| \\ &= |2^{h_t} - 2^{h^*}| \\ &\rightarrow 0, \end{aligned}$$

since the function $h \mapsto 2^h$ is continuous.

- (vi) Let (X, d) be a non-empty compact metric space, and consider any continuous utility function $u : X \rightarrow \mathbb{R}$. Let X^* be the set of optimal choices, i.e. $X^* = \operatorname{argmax}_{x \in X} u(x)$. Prove that X^* is non-empty and compact.

Answer. By the extreme value theorem, u has a maximum value, \bar{u} . This means that $X^* = u^{-1}(\{\bar{u}\})$ is non-empty. Moreover, since $\{\bar{u}\}$ is a finite set, it is closed. Since u is continuous, $X^* = u^{-1}(\{\bar{u}\})$ is closed. Finally, since X^* is a closed subset of a compact set X , we conclude X^* is compact.

- (vii) Consider the optimisation problem:

$$\begin{aligned} \max_{a, b \in \mathbb{R}_+} & p(x - a) + (1 - p)(y - b) \\ \text{s.t.} & u(a, e) \geq u(b, 0) \text{ and } v(b, 0) \geq v(a, e), \end{aligned}$$

where u and v are continuous functions, $p \in [0, 1]$ and $x, y \in \mathbb{R}_+$. Assume that there is a feasible choice (\bar{a}, \bar{b}) that satisfies both constraints. Prove that there exists an optimal choice (a^*, b^*) .

The economic content of this model – which is not necessary for solving the problem – is as follows. A proportion p of the workers are “good”, i.e. they have utility function u , not v , and they produce output x , not y . A recruiter wants to hire an optimal mix of good and bad workers. But he can’t tell them apart. Instead, students can put effort e into their education, which the recruiter can observe. So, the recruiter selects wages a for the highly educated and b for lowly educated students.

Comment. The original exam question left out the key assumption that there is a feasible choice. I was generous with students who had the right approach in applying the Extreme Value Theorem.

Answer. Let $X = \mathbb{R}_+^2$ be the set of all possible (a, b) values, and let

$$D = \{(a, b) \in X : u(a, e) \geq u(b, 0), v(b, 0) \geq v(a, e)\}$$

be the subset that satisfies both constraints. Since $(\bar{a}, \bar{b}) \in D$, the supremum gives a value of at least $Y = p(x - \bar{a}) + (1 - p)(y - \bar{b})$. This means we can add an extra (redundant) constraint, that says you may not spend more than Y on a or b , i.e.

$$D' = \{(a, b) \in D : pa \leq Y, (1 - p)b \leq Y\}.$$

Now, D' is bounded because $D' \subseteq [0, Y]^2$. I now show that D' is closed. Let $f(a, b) = u(a, e) - u(b, 0)$ and $g(a, b) = v(b, 0) - v(a, e)$. These are both continuous functions. Now, $D' = f^{-1}(\mathbb{R}_+) \cap g^{-1}(\mathbb{R}_+) \cap [0, Y]^2$ is the intersection of three closed sets; the first of these is closed because f is continuous and \mathbb{R}_+ is a closed set. Since D' is closed and bounded, it follows that D' is compact by the Bolzano-Weierstrass theorem.

Since the objective is continuous, by the extreme value theorem it has a maximum in D' , and hence on D (since every point in $D \setminus D'$ is inferior to some point in D').

- (viii) Consider the Bellman equation of a firm that makes a profit of $\pi(n)$ when it has $n \in \mathbb{N}$ workers, but it costs $\Delta(n, n')$ to hire (or fire) $n' - n$ workers tomorrow when it has n workers today:

$$V(n) = \sup_{n' \in \mathbb{N}} \pi(n) - \Delta(n, n') + \beta V(n').$$

Assume that π is bounded, $\Delta(n, n) = 0$, $\Delta(n, n') \geq 0$ and $\Delta(n, \cdot)$ is unbounded. Prove that the Bellman equation has exactly one solution.

Comment. Most students focused on establishing that the Bellman operator is a contraction, neglecting the other elements of the answer. It is also important to say what the domain and co-domain of the Bellman operator is, and check that it is a complete metric space.

Answer. Consider the Bellman operator,

$$T(V)(n) = \sup_{n' \in \mathbb{N}} \pi(n) - \Delta(n, n') + \beta V(n').$$

Now, V^* solves the Bellman equation if and only if $T(V^*) = V^*$. So it suffices to prove that T has a unique fixed point. I will do this by proving that T is a contraction on the complete metric space $(B(\mathbb{N}), d_\infty)$ and applying Banach's fixed point theorem. (Recall that $(B(X), d_\infty)$ is a complete metric space, regardless of the domain X .)

I now prove that T is a contraction. Let $f(n)$ be the optimal choice of n' when the state is n and tomorrow's value function is V . Then

$$\begin{aligned} T(V)(n) &= \pi(n) - \Delta(n, f(n)) + \beta V(f(n)) \\ &= \pi(n) - \Delta(n, f(n)) + \beta W(f(n)) + \beta V(f(n)) - \beta W(f(n)) \\ &\leq \sup_{n' \in \mathbb{N}} [\pi(n) - \Delta(n, n') + \beta W(n')] + \beta V(f(n)) - \beta W(f(n)) \\ &= T(W)(n) + \beta V(f(n)) - \beta W(f(n)) \\ &\leq T(W)(n) + \beta d_\infty(V, W), \end{aligned}$$

so $T(V)(n) - T(W)(n) \leq \beta d_\infty(V, W)$ for all n .

Swapping the roles of V and W in the algebra above gives $T(W)(n) - T(V)(n) \leq \beta d_\infty(W, V)$ for all n . Therefore,

$$|T(V)(n) - T(W)(n)| \leq \beta d_\infty(V, W)$$

for all n . Taking suprema of both sides gives $d_\infty(T(W), T(V)) \leq \beta d_\infty(W, V)$. So T is a contraction of degree β as required.

30: Micro 1, May 2018

Suppose that Idaho farmers each own a field of Russet potatoes, and North Carolina farmers each own a field of sweet potatoes. Assume there are an equal number of farmers in Idaho and North Carolina. All farmers have the same preferences. In this question, do *not* assume that potatoes are (for all prices) normal goods.

- (i) Formulate a pure-exchange competitive model of the sweet potato and Russet potato markets.

Comment. This part of the question is particularly easy, so most students answered it well. The most common mistake was not introducing separate notation to capture the different choices and endowments of Idaho and North Carolina farmers.

Many students included firms and production functions, even though the question explicitly said “pure-exchange.” This overcomplicates the answer (and often lead to mistakes), but isn’t a problem in itself.

Answer.

Farmer’s problem. Each farmer has a type $t \in \{ID, NC\}$, and there are N of each type. A farmer of type t is endowed with $e^t = (e_r^t, e_s^t)$ units of Russet and sweet potatoes, respectively. Specifically, $e_r^{ID} = e_s^{NC} = 1$ and $e_s^{ID} = e_r^{NC} = 0$. Russet potatoes trade at price p_s , and sweet potatoes at price p_r . Each farmer of type t sell their endowment e^t , and purchases $x^t = (x_r^t, x_s^t)$ for consumption. This is chosen to maximise the utility function $u(x^t)$. The utility maximisation problem is

$$\begin{aligned} \max_{x^t} u(x^t) \\ \text{s.t. } p \cdot x^t = p \cdot e^t. \end{aligned}$$

Equilibrium. Prices (p_r, p_s) and quantities $(x_r^{ID}, x_s^{ID}, x_r^{NC}, x_s^{NC})$ form an equilibrium if the quantities solve the farmer’s problems above, and the potato markets clear:

$$\begin{aligned} N e_r^{ID} &= N x_r^{ID} + N x_r^{NC} \\ N e_s^{NC} &= N x_s^{ID} + N x_s^{NC}. \end{aligned}$$

- (ii) Use dynamic programming to reformulate the farmers’ utility maximization problem. Specifically, write a Bellman equation that connects the indirect utility function (that gives each farmer’s value as a function of prices and endowments) to the expenditure function (that gives each farmer’s net expenditure as a function of prices, endowments, and a utility target).

Comment. This question had a mistake in it: the expenditure function gives money spent on consumption, not consumption quantities. Most students were able to formulate the expenditure function, but many students had difficulty with the Bellman equation.

Answer. Let $V(p, e^t)$ be the indirect utility function, i.e.

$$\begin{aligned} V(p, e^t) &= \max_{x^t} u(x^t) \\ \text{s.t. } & p \cdot x^t = p \cdot e^t. \end{aligned}$$

Let $E(p, e^t, \bar{u})$ be the expenditure function, i.e.

$$\begin{aligned} E(p, e^t, \bar{u}) &= \min_{x^t} p \cdot (x^t - e^t) \\ \text{s.t. } & u(x^t) \geq \bar{u}. \end{aligned}$$

The Bellman equation relating these two value functions is

$$\begin{aligned} V(p, e^t) &= \max_{\bar{u}} \bar{u} \\ \text{s.t. } & E(p, e^t, \bar{u}) \leq 0. \end{aligned}$$

- (iii) Suppose that at a particular price level, Idaho farmers respond to a price increase in Russet potatoes by consuming more Russet potatoes. Prove that this implies that Russet potatoes are normal goods for Idaho farmers. *Recall that a good X is an inferior good if the demand for X decreases when the wealth of the consumer increases. Hint 1: apply the envelope theorem to the expenditure function. Hint 2: you may find the Slutsky equation from the lecture notes helpful.*

$$\underbrace{\frac{\partial x_i(p, m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p, u)}{\partial p_j} \right]_{u=v(p, m)}}_{\text{substitution effect}} + \underbrace{-x_j(p, m) \frac{\partial x_i(p, m)}{\partial m}}_{\text{wealth lost income effect}}.$$

Comment. The original question asked the opposite: if the price increase leads to more Russet potato consumption, then this implies that Russet potatoes are inferior goods for Idaho farmers. The question was wrong – it depends on model parameters. (I forgot a minus sign.)

The crux of this question is: when the price of Russet potatoes increases, then Russet potato sellers in Idaho become wealthier (not poorer, as in the textbook analysis of Giffen goods).

In any case, I wanted to see a proof that the substitution effect is negative.

Answer. Suppose p^* is a price vector in this situation. I will show below that the Slutsky equation in this problem is:

$$\frac{\partial x_r(p^*, m)}{\partial p_r} = \left[\frac{\partial h_r(p^*, \bar{u})}{\partial p_r} \right]_{\bar{u}=V(p^*, m)} + (e_r^{ID} - x_r(p^*, m)) \frac{\partial x_r(p^*, m)}{\partial m}.$$

The question states that the left side is greater than zero. I will show below that the first term on the right side (the substitution effect) is negative, so the second term on the right side (the income effect) must be positive. Now, from the budget

constraint I can deduce that $e_r^{ID} - x_r(p^*, m) > 0$ when $m = p^* \cdot e^{ID}$. So I conclude that the income effect

$$\frac{\partial x_r(p^*, m)}{\partial m} > 0$$

and that Russet potatoes are a normal good at Idaho farmers' wealth level $m = p^* \cdot e^{ID}$.

Step 1. To establish this version of the Slutsky equation, I start with the identity

$$h_r(p, \bar{u}) = x_r(p, E(p, e, \bar{u})).$$

Differentiating both sides with respect to p_r gives

$$\frac{\partial h_r(p, \bar{u})}{\partial p_r} = \left[\frac{\partial x_r(p, m)}{\partial p_r} + \frac{\partial x_r(p, m)}{\partial m} \frac{\partial E(p, e, \bar{u})}{\partial p_r} \right]_{m=p \cdot e}.$$

By the envelope theorem,

$$\frac{\partial E(p, e, \bar{u})}{\partial p_r} = h_r(p, \bar{u}) - e_r.$$

Substituting and rearranging gives the Slutsky equation above.

Step 2. I know show that the term

$$\frac{\partial h_r(p^*, \bar{u})}{\partial p_r}$$

is negative. By the envelope theorem (again),

$$\frac{\partial E(p^*, 0, \bar{u})}{\partial p_r} = h_r(p^*, \bar{u}) - e_r.$$

Differentiating again gives

$$\frac{\partial^2 E(p^*, 0, \bar{u})}{\partial p_r^2} = \frac{\partial h_r(p^*, \bar{u})}{\partial p_r},$$

so that both sides equal the term of interest (the substitution effect). So it remains to show that the second derivative of the expenditure function $E(p^*, 0, \bar{u})$ with respect to the price of Russet potatoes p_r is negative. Now, $E(\cdot, 0, \bar{u})$ is the lower envelope of linear functions, one for each possible consumption choice x . Therefore E is concave. We conclude that differentiating twice with respect to p_r gives a negative number, i.e. the substitution effect is negative.

- (iv) *For the rest of this question, suppose that your model has two equilibria (while holding all model parameters fixed, including endowments): one in which both types of farmer consume the same things, and one in which North Carolina farmers consume more of both types of potato than Idaho farmers.*

Which of these two equilibria do North Carolina farmers prefer?

Comment. The actual exam question was somewhat ambiguous; I had written “North Carolina farmers consume more of both types of potato.” In any case, it makes little difference to the answer.

Answer. North Carolina farmers prefer the second equilibrium.

Since this is a pure-exchange economy, the fact that the North Carolina farmers consume more of both types of potatoes than Idaho farmers implies that North Carolina farmers consume more than half of both types of potatoes. Therefore, they consume more of each type of potato in the second equilibrium than the first equilibrium (where they consume exactly half of each type). Since farmers prefer to eat more potatoes, I conclude that they prefer the second equilibrium.

- (v) Sketch and explain a graph showing a possible shape of the excess demand function of sweet potatoes as a function of the price of sweet potatoes.

Answer. There are many possible shapes, but every graph must have the following features:

- As $p_s \rightarrow 0$, excess demand approaches ∞ . When $p_s = 0$, North Carolina farmers demand an infinite amount of sweet potatoes, but can not afford any Russet potatoes. Idaho farmers consume their own Russet potatoes and an infinite amount of sweet potatoes.
- As $p_s \rightarrow \infty$, excess demand approaches zero or a negative number. As p_s increases sufficiently, Idaho farmers’ demand for sweet potatoes goes towards zero. North Carolina farmers can afford at most to consume their sweet potato endowments. Adding up, excess demand for sweet potatoes is either zero or negative when p_s is large.
- Since there are at least two equilibria, the excess demand function must cross zero at least twice. In fact, it must cross at least three times because it starts above and finishes below the horizontal axis (unless it touches the horizontal axis without crossing it).

- (vi) For each of the two equilibria, devise a lump-sum tax policy that implements that equilibrium.

Answer. Both policies are the same: no taxes. By assumption, both equilibria are already equilibria at the given endowments.

- (vii) * Suppose (X, d_X) and (Y, d_Y) are non-empty metric spaces, and consider the metric space $(Z, d_Z) = (X \times Y, d_Z)$ where

$$d_Z(x, y; x', y') = \max \{d_X(x, x'), d_Y(y, y')\}.$$

Prove that if (Z, d_Z) is a compact metric space, then (X, d_X) is a compact metric space.

Answer. Let x_n be any sequence in (X, d_X) . Since (Y, d_Y) is non-empty, it has some point inside it, $\bar{y} \in Y$. Consider the sequence $z_n = (x_n, \bar{y})$.

Since (Z, d_Z) is compact, z_n has a convergent subsequence z_{n_k} , where $z_{n_k} \rightarrow z^*$. We can write $z^* = (x^*, y^*)$. (In fact $y^* = \bar{y}$, although this is not relevant.) Since

$d_Z(z_{n_k}, z^*) \rightarrow 0$ and $d_X(x_{n_k}, x^*) \leq d_Z(z_{n_k}, z^*)$, it follows that $d_X(x_{n_k}, x^*) \rightarrow 0$. We deduce that $x_{n_k} \rightarrow x^*$. Therefore, x_{n_k} has a convergent subsequence. We conclude that (X, d_X) is a compact metric space.

- (viii) * Let (X, d) be a complete metric space, and let $A \subseteq X$. Suppose that $f : X \rightarrow X$ is a contraction, and that $f(A) \subseteq A$. Prove that f has a fixed point x^* that lies in the closure of A .

Answer. Let \bar{A} be the closure of A . I first show that $f(\bar{A}) \subseteq \bar{A}$. Pick any $a^* \in \bar{A}$. Then there must be some sequence $a_n \in A$ with $a_n \rightarrow a^*$. By assumption, each $b_n = f(a_n) \in A$. Since f is (Lipshitz) continuous $b_n \rightarrow f(a^*)$. Since \bar{A} is closed, this means $f(a^*) \in \bar{A}$. I conclude that $f(\bar{A}) \subseteq \bar{A}$.

Now, since \bar{A} is a closed subset of a complete metric space, (\bar{A}, d) is a complete metric space. Since f is a contraction on the complete metric space (\bar{A}, d) , it has a unique fixed point x^* that lies inside \bar{A} .

31: AME, December 2018

Part A

There are two car factories, Alfa and Buggy, both of which use machines and labour to make (identical) cars. Assume that Alfa is half as productive, i.e. at the same input levels, it produces half as many cars. Households are endowed with labour and machines, which they rent out to the factories. Households consume cars and leisure.

- (i) Write down a competitive equilibrium model of the car, labour and machine markets.

Answer. Households. There are N identical households. Each household is endowed with m machines which it rents at price r , and 24 hours of time, of which it supplies h to the labour market at price w . The household also receives a dividend of Π/N , where $\Pi = \pi_A(p; r, w) + \pi_B(p; r, w)$ are defined below. The household purchases c cars at price p . This gives the household a utility of $u(c, 24 - h)$. The household's problem is

$$\begin{aligned} \max_{c, h} u(c, 24 - h) \\ \text{s.t. } pc = rm + wh + \Pi/N. \end{aligned}$$

Firms. There are two firms, $i \in A, B$, corresponding to Alfa and Buggy. Firm i hires H_i hours of labour and rents M_i machines to produce $C_i = f_i(H_i, M_i)$ cars. Its profit function is

$$\pi_i(p; r, w) = \max_{H_i, M_i} pf_i(H_i, M_i) - wH_i - rM_i.$$

Equilibrium. An equilibrium consists of prices (p, r, w) and quantities (c, h, m) and $(C_i, H_i, M_i)_{i \in \{A, B\}}$, such that all choices solve the respective problems above, and all three markets clear, i.e.

$$\begin{aligned} Nc &= C_A + C_B \\ Nh &= H_A + H_B \\ Nm &= M_A + M_B. \end{aligned}$$

- (ii) Suppose Alfa and Buggy merge into a single firm owning the two factories. Write down the merged firm's profit function as a Bellman equation involving the individual firms' profit functions.

Comment. A common mistake was to try to put an optimisation problem in, where none was required.

Answer. The combined value function would be

$$\pi(p; w, r) = \pi_A(p; w, r) + \pi_B(p; w, r).$$

- (iii) Prove that the merged firm's profit function is convex.

Answer. Since the sum of convex functions is convex, this amounts to proving that π_A and π_B are convex functions. Recall

$$\pi_i(p; r, w) = \max_{H_i, M_i} pf_i(H_i, M_i) - wH_i - rM_i.$$

Now, for each choice of (H_i, M_i) , the function $(p; r, w) \mapsto pC_i - wH_i - rM_i$ mapping prices to profits is linear, where $C_i = f_i(H_i, M_i)$. Therefore π_i is the upper envelope of linear (and hence convex) functions. So π_i is convex.

(iv) Prove that Alfa produces fewer cars.

Comment. Many students tried to use first-order conditions, although they are not helpful here.

Answer. If we define

$$\pi^*(z; p; r, w) = \max_{H, M} pz f(H, M) - wH - rM$$

then $\pi_A(p; r, w) = \pi^*(\frac{1}{2}; p; r, w)$ and $\pi_B(p; r, w) = \pi^*(1; p; r, w)$.

By the envelope theorem,

$$\frac{\partial \pi^*}{\partial z} = pf(H, M) = pC(z; p; r, w),$$

where $C(z; p; r, w)$ is the firm's supply when it has productivity parameter z and faces prices (p, r, w) .

Since π^* is convex in z (by a similar argument as above), both sides are increasing in z . We conclude that output is increasing in productivity, so Alfa produces fewer cars.

Part B

(i) Consider any metric space (X, d) , and any closed ball $A = B_r(x) = \{y \in X : d(x, y) \leq r\}$ with centre $x \in X$ and radius $r > 0$. Prove that A is a closed set.

Comment. A common mistake was to write that the boundary of A is $\partial A = \{y \in X : d(x, y) = r\}$. But this is false. For example, consider the metric space $([-1, 1], d_2)$. In this space, the boundary of the closed ball $B_1(0)$ is the empty set, not $\{-1, 1\}$.

Answer. Let $a_n \in A$ be a convergent sequence with $a_n \rightarrow a^*$. We would like to prove that $a^* \in A$.

Since $a_n \in A$, we know that $d(a_n, x) \leq r$. By the triangle inequality,

$$d(a^*, x) \leq d(x, a_n) + d(a_n, a^*) \leq r + d(a_n, a^*) \text{ for all } n.$$

Since $a_n \rightarrow a^*$, for every $s > 0$ there exists some N such that $d(a_n, a^*) < s$. This implies that

$$d(a^*, x) \leq r + s$$

for all $s > 0$. We conclude that $d(a^*, x) \leq r$ and hence $a^* \in A$.

- (ii) Find a counter-example to the following statement: for every set A , the interior of the boundary of A is empty.

Answer. Let $(X, d) = (\mathbb{R}, d_2)$ and $A = \mathbb{Q}$. Now, $\partial A = \mathbb{R}$, which is the whole space, and is therefore open. Since ∂A is an open set, $\text{int}(\partial A) = \partial A$. We conclude that $\text{int}(\partial A) = \mathbb{R}$ is not empty.

- (iii) Recall the metric space $(B[0, 1], d_\infty)$, where $B[0, 1]$ is the set of bounded functions from $[0, 1]$ to \mathbb{R} and $d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$. Consider the set $A = \{f \in B[0, 1] : f(x) > 0\}$ and the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x > 0. \end{cases}$$

Prove or disprove that f is a boundary point of A .

Comment. Many students wrote that the complement of A is $B = \{f \in B[0, 1] : f(x) \leq 0\}$. This is mistaken, for example $g(x) = x - \frac{1}{2}$ is not in A , so it should be in the complement.

Answer. The function f is a boundary point of A .

First, notice that $f \in A$, so the trivial sequence $a_n = f$ satisfies the properties that $a_n \rightarrow f$ and $a_n \in A$.

Second, consider the sequence $b_n(x) = f(x) - 1/n$. This sequence lies outside of A since $b_n(1/n) = 0$. Moreover, $d_\infty(b_n, f) = 1/n$, so $b_n \rightarrow f$.

We conclude that f is a boundary point of A .

- (iv) Find a metric d such that (\mathbb{Q}, d) is a complete metric space. Recall that \mathbb{Q} is the set of rational numbers, i.e. ratios of whole numbers.

Answer. Let d be the discrete metric, $d(x, y) = I(x \neq y)$. Then (\mathbb{Q}, d) is a complete metric space.

To prove this, let $x_n \in \mathbb{Q}$ be any Cauchy sequence. We need to establish that x_n is a convergent sequence. Since x_n is a Cauchy sequence, there exists a number N such that

$$d(x_n, x_m) < 1 \text{ for all } n, m \geq N.$$

But $d(x_n, x_m) < 1$ if and only if $x_n = x_m$. Therefore, $x_n = x_N$ for all $n > N$. We conclude that $x_n \rightarrow x_N$.

- (v) Prove that $(CB[0, 1], d_1)$ is a metric space. Recall that

$$CB[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R}, f \text{ is continuous and bounded}\}$$

and $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.

Answer. We prove the three properties in turn:

- $f = g$ if and only if $d_1(f, g) = 0$. First, $d_1(f, f) = \int_0^1 |f(x) - f(x)| dx = \int_0^1 0 dx = 0$.

Conversely, suppose $d_1(f, g) = 0$. Let $h(x) = |f(x) - g(x)|$ where $h : [0, 1] \rightarrow \mathbb{R}_+$. So $d_1(f, g) = \int_0^1 h(x) dx = 0$. It suffices to show that $h(x) = 0$ for all $x \in [0, 1]$. Suppose for the sake of contradiction that $h(x^*) > 0$ for some $x^* \in [0, 1]$. Let $y^* = h(x^*)$. Consider then open ball $N_r(y^*)$ of radius $r = y^*/2$. Since h is continuous, the open ball characterisation of continuity implies that there exists an open ball $N_s(x^*)$ such that $h(N_s(x^*)) \subseteq N_r(y^*)$. This implies that

$$\int_{x^*-s}^{x^*+s} h(x) dx \geq \int_{x^*-s}^{x^*+s} (y^* - r) dx = (2s)(y^* - y^*/2) = sy^* > 0.$$

But the left side is smaller than $\int_0^1 h(x) dx$, so this contradicts $\int_0^1 h(x) dx = 0$. Therefore, the assumption $h(x^*) > 0$ for some x^* is false. We conclude that $h(x) = 0$ and hence $f(x) = g(x)$ for all $x \in [0, 1]$.

- $d_1(f, g) = d_1(g, f)$. Expanding the definitions gives

$$\begin{aligned} d_1(f, g) &= \int_0^1 |f(x) - g(x)| dx \\ &= \int_0^1 |g(x) - f(x)| dx \\ &= d_1(g, f), \end{aligned}$$

as required.

- $d_1(f, h) \leq d_1(f, g) + d_1(g, h)$. Expanding the definitions gives

$$\begin{aligned} d_1(f, h) &= \int_0^1 |f(x) - h(x)| dx \\ &= \int_0^1 |f(x) - g(x) + g(x) - h(x)| dx \\ &\leq \int_0^1 (|f(x) - g(x)| + |g(x) - h(x)|) dx \\ &= d_1(f, g) + d_1(g, h), \end{aligned}$$

as required.

- (vi) Construct an open cover of $(0, 1)$ inside the metric space (\mathbb{R}, d_2) that has no finite sub-cover.

Answer. Consider the sets $\mathcal{U} = \{(1/(n+1), 1) : n \in \mathbb{N}\}$, or equivalently $\mathcal{U} = \{\emptyset, (1/2, 1), (1/3, 1), \dots\}$. First, \mathcal{U} covers $(0, 1)$, since every number $x > 0$ is greater than some rational number p/q , which is greater than $1/q$. Second, \mathcal{U} is an open cover, since it consists of open balls. Finally, \mathcal{U} does not have a finite sub-cover. Any finite subset of $\mathcal{U}^* \subseteq \mathcal{U}$ has a largest set with left end-point $1/(N+1)$. None of the sets in \mathcal{U}^* contain $1/(N+2)$.

(vii) Prove the following version of Banach's fixed point theorem:

Let (X, d) be a compact metric space. If $f : X \rightarrow X$ satisfies the property $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, then f has a unique fixed point.

Hint: $\min_{x \in X} d(x, f(x))$.

Comment. This theorem is adapted Border and Aliprantis (2005), Section 3.12.

No students got this right. Most students ignored the hint and attempted to apply Banach's fixed point theorem. But Banach's fixed point theorem is inapplicable because we only know f is Lipschitz continuous of degree 1. (Contractions must have a Lipschitz degree of strictly less than 1.) One student pursued another promising proof strategy, but left some major holes.

Answer. Uniqueness. Suppose for the sake of contradiction that $x^* = f(x^*)$ and $x^{**} = f(x^{**})$ where $x^* \neq x^{**}$. Then $d(f(x^*), f(x^{**})) = d(x^*, x^{**})$. But this contradicts the assumption that $d(f(x^*), f(x^{**})) < d(x^*, x^{**})$.

Existence. Let $g(x) = d(x, f(x))$. Now, $g : X \rightarrow \mathbb{R}$ is a continuous function, because d and f are continuous. (d is continuous by a homework question, and f is continuous because it is Lipschitz continuous of degree 1.) Since the domain is compact, there is a point $x^* \in X$ that minimises g . We claim that x^* is a fixed point. Suppose for the sake of contradiction that $g(x^*) > 0$. The assumption on f implies that $g(f(x^*)) = d(f(x^*), f(f(x^*))) < d(x^*, f(x^*)) = g(x^*)$. This contradicts x^* minimising g .

(viii) Let $a \in \mathbb{R}_+$ be assets, $e \in \{0, 1\}$ be employment status (0 being unemployed), $c \in \mathbb{R}_+$ be consumption, $u(c)$ be the utility of consuming c , w be the wage, $p(0)$ be the probability of finding a job, and $p(1)$ be the probability of keeping a job. Consider the Bellman equation,

$$V(a, e) = \max_{c, a'} u(c) + \beta[p(e)V(a', 1) + (1 - p(e))V(a', 0)]$$

$$\text{s.t. } c + a' = a + we.$$

Suppose that u is concave and bounded.

- Define the Bellman operator, including the metric spaces for the domain and co-domain.
- Prove that the Bellman operator is a contraction.
- Prove that the Bellman equation has a unique bounded solution V^* .
- Prove that V^* is strictly concave in a .

Comment. There was a mistake in the exam question: the objective included a rather than a' . Some students noticed the mistake, but did not contact me during the exam (via the invigilators) to alert everyone.

Answer.

- (a) Let the set of possible state variables be $X = \mathbb{R}_+ \times \{0, 1\}$, with distances measured by d_1 . The Bellman operator is a function $T : B(X) \rightarrow B(X)$ defined by

$$T(V)(a, e) = \sup_{a'} u(a + we - a') + \beta[p(e)V(a', 1) + (1 - p(e))V(a', 0)].$$

Distances on $B(X)$ are measured by d_∞ .

- (b) We adapt the version of Blackwell's lemma in the notes to this situation. Note that we assumed that u is bounded. So if $V \in B(X)$, then $T(V)$ exists and lies in $B(X)$.

Now consider two functions $V, W \in B(X)$. Let $a'(a, e)$ be an optimal choice when the value function (the input into the Bellman operator) is V . (Note: there might not be an optimal choice – for simplicity we show the proof assuming there is an optimal choice; dropping this assumption requires messier notation involving sup, but the same logic.) Using the shorthand $E[V(a', e')] = p(e)V(a', 1) + (1 - p(e))V(a', 0)$, we have

$$\begin{aligned} T(V)(a, e) &= u(a + we - a'(a, e)) + \beta E[V(a'(a, e), e')] \\ &= u(a + we - a'(a, e)) + \beta E[W(a'(a, e), e')] - \beta E[W(a'(a, e), e')] + \beta E[V(a'(a, e), e')] \\ &\leq \left[\sup_{\hat{a}'} u(a + we - \hat{a}') + \beta E[W(\hat{a}', e')] \right] - \beta E[W(a'(a, e), e')] + \beta E[V(a'(a, e), e')] \\ &= T(W)(a, e) - \beta E[W(a'(a, e), e')] + \beta E[V(a'(a, e), e')]. \end{aligned}$$

Rearranging, we have

$$T(V)(a, e) - T(W)(a, e) = \beta E[V(a'(a, e), e') - W(a'(a, e), e')] \leq \beta d_\infty(V, W).$$

Since this is true for all (a, e) , we find that:

$$\sup_{(a, e) \in X} [T(V)(a, e) - T(W)(a, e)] \leq \beta d_\infty(V, W).$$

Swapping the role of V and W , we arrive at:

$$\sup_{(a, e) \in X} [T(W)(a, e) - T(V)(a, e)] \leq \beta d_\infty(W, V).$$

We conclude that $d_\infty(T(V), T(W)) \leq \beta d_\infty(V, W)$, so T is a contraction of degree β .

- (c) The (co)-domain of T is $(B(X), d_\infty)$. Now, the co-domain of the functions in $B(X)$ is (\mathbb{R}, d_2) , which is a complete metric space. So a theorem in the notes established that $(B(X), d_\infty)$ is complete. Therefore, T is a contraction on a complete metric space. Banach's fixed point theorem establishes that T has a unique fixed point, V^* .

- (d) First, we establish that V^* is (weakly) concave. Consider the metric space (A, d_∞) , where

$$A = \{f \in B(X) : f \text{ is concave}\}.$$

We first check that T is a self-map on A . Recall that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if and only if for all $x, y \in \mathbb{R}^n$ and all $t \in [0, 1]$,

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y).$$

We will use this theorem to check if $T(V)$ is concave in a , assuming u and V are concave:

$$\begin{aligned} & T(V)(ta + (1 - t)\tilde{a}, e) \\ &= u(a + we - a'(ta + (1 - t)\tilde{a}, e)) + \beta E[V(a'(ta + (1 - t)\tilde{a}, e), e')] \\ &\geq tu(a + we - a'(a, e)) + (1 - t)u(a + we - a'(\tilde{a}, e)) + \beta E[V(a'(ta + (1 - t)\tilde{a}, e), e')] \\ &\geq tu(a + we - a'(a, e)) + (1 - t)u(a + we - a'(\tilde{a}, e)) \\ &\quad + \beta E[tV(a'(a, e), e') + (1 - t)V(a'(\tilde{a}, e), e')] \\ &= t[u(a + we - a'(a, e)) + \beta E[V(a'(a, e), e')]] \\ &\quad + (1 - t)[u(a + we - a'(\tilde{a}, e)) + \beta E[V(a'(\tilde{a}, e), e')]] \\ &= tT(V)(a, e) + (1 - t)T(V)(\tilde{a}, e). \end{aligned}$$

In a homework question, we established that (A, d_∞) is complete. Therefore $T : A \rightarrow A$ is a contraction on a complete metric space. We conclude that the fixed point V^* (the same one as before) must lie inside A .

Finally, the above inequalities are strict if u and V are strictly concave in a . This means that T maps concave functions to strictly concave functions. Therefore, $T(V^*)$ is strictly concave. But $T(V^*) = V^*$, so V^* is strictly concave.

32: Micro 1, December 2018

According to EU regulation 543/2011, apples marketed as “class I, colour group A apples” must have “1/2 of total surface red coloured,” whereas class II apples have no colour requirements.

Suppose that a farm make class I and class II apples out of labour. The farm can not control what fraction of apples are of each class, only the total number of apples grown. A beverage firm makes apple juice out of labour and apples – the two classes of apple are perfect substitutes as far as the firm is concerned. Households sell labour and buy both types of apples and apple juice; they prefer class I apples over class II apples.

- (i) Formulate a competitive model of the labour, apple, and apple juice markets.

Comment. There were many problems with formulating the labour market. The question did not specify if working in the farm is more tiring than working in the juice factory. It is ok to assume that labour is supplied inelastically (as I did below) to a single labour market. However, it is *not* ok to assume that workers split their labour between the two firms in exogenous portions. If you choose to formulate the model in which the workers choose which firms to work for, then this must reflect prices and preferences. For example, if the workers are indifferent (in terms of the utility function), then they would choose the firm offering the highest wage.

Another common mistake was forgetting to include the beverage firm’s demand for apples in the apple market clearing conditions.

Most students assumed that the two classes of apples occur in constant proportions, regardless of the size of the farm. This is ok, but it is an unnecessary assumption – see the sample solution below for a simpler and more general approach.

Answer. Households supply labour ℓ at price w . They receive dividends π/n , where $\pi = \pi^f + \pi^b$ is the firms’ profits (see below) and n is the number of households. They demand the two classes of apples, a_1 and a_2 at prices p_1 and p_2 . They demand j units of apple juice at price p_j . This gives them utility $u(\ell, a_1, a_2, j)$. The utility maximisation problem is

$$\begin{aligned} \max_{\ell, a_1, a_2, j} & u(\ell, a_1, a_2, j) \\ \text{s.t.} & p_1 a_1 + p_2 a_2 + p_j j = w\ell + \pi/n. \end{aligned}$$

Farm. The farm hires L_f workers and produces $A_1 = f_1(L_f)$ and $A_2 = f_2(L_f)$ of the two classes of apples. Its profit function is

$$\pi^f(p_1, p_2; w) = \max_{L_f} p_1 f_1(L_f) + p_2 f_2(L_f) - wL_f.$$

Beverage firm. The firm hires L_b workers and buys A_{b1} and A_{b2} of the two classes of apples to make $J = g(L_b, A_{b1} + A_{b2})$ units of juice. Its profit function is

$$\pi^b(p^j; p_1, p_2, w) = \max_{L_b, A_{b1}, A_{b2}} p_j g(L_b, A_{b1} + A_{b2}) - wL_b - p_1 A_{b1} - p_2 A_{b2}.$$

Equilibrium. An equilibrium consists of prices (w, p_1, p_2, p_j) and quantities

$$(\ell, a_1, a_2, j, L_f, A_1, A_2, L_b, A_{b1}, A_{b2}, J)$$

such that the quantities are solutions to the respective optimisation problems above, and the markets clear:

$$\begin{aligned} n\ell &= L_f + L_b \\ na_1 + A_{b1} &= A_1 \\ na_2 + A_{b2} &= A_2 \\ nj &= J. \end{aligned}$$

- (ii) Prove that if the beverage firm buys both types of apples, then the two types of apples trade at the same price.

Comment. A common mistake was to write g' (or similar) to represent a marginal productivity. But this notation is inadequate, because there are multiple factors of production, and it is unclear which factor is being increased.

Many students were rather vague about the role of perfect substitutes – this becomes quite simple to see if this is correctly translated into mathematics by writing $A_{b1} + A_{b2}$ or similar.

Answer. The beverage firm's first-order conditions with respect to A_{b1} and A_{b2} are

$$\begin{aligned} p_j D_2 g(L_b, A_{b1} + A_{b2}) &= p_1 \\ p_j D_2 g(L_b, A_{b1} + A_{b2}) &= p_2. \end{aligned}$$

The left side is equal in both cases, so $p_1 = p_2$.

Note: the first-order conditions can be written with different notation like this:

$$p_j \left. \frac{\partial g(L, A)}{\partial A} \right|_{L=L_b, A=A_{b1}+A_{b2}} = p_1.$$

- (iii) Suppose the farm and the beverage firm merge into a single firm. Formulate the merged firm's profit function.

Comment. One common mistake was to assume that the merged firm only sells apple juice.

Answer.

$$\begin{aligned} \pi(w, p_1, p_2, p_j) \\ = \max_{L_f, L_b, A_{b1}, A_{b2}} p_1 [f_1(L_f) - A_{b1}] + p_2 [f_2(L_f) - A_{b2}] + p_j g(L_b, A_{b1} + A_{b2}) - w[L_f + L_b]. \end{aligned}$$

- (iv) Suppose at some equilibrium, the merged firm uses some class I apples for apple juice production. Prove that if the price of class I apples decreases (to a non-equilibrium price), then the merged firm responds by allocating more apples to apple juice production.

Comment. This problem is difficult to solve without dynamic programming – the merged firm’s problem is complicated, but since the merged firm makes the same decisions as the individual firms, we can study the beverage firm alone. Few students realised this.

Answer. First, it can be shown that the principle of optimality holds, i.e. the following Bellman equation is satisfied by the profit functions defined above:

$$\pi(w, p_1, p_2, p_j) = \pi^f(p^j; p_1, p_2, w) + \pi^b(p^j; p_1, p_2, w).$$

Therefore, the juice decision of the merged firm is the same as that of the beverage firm.

Since the firm uses some class I apples for apple juice production, we know that $p_1 = p_2$. (See the previous question.)

Since the juice production technology works equally well with both classes of apple, we can reformulate the merged firms profit function as

$$\pi(w, p_1, p_2, p_j) = \pi^f(p^j; p_1, p_2, w) + \pi^b(p^j; \min\{p_1, p_2\}, w).$$

where

$$\pi^b(p^j; p, w) = \max_{L_b, A_b} p_j g(L_b, A_b) - wL_b - pA_b.$$

By the envelope theorem,

$$\frac{\partial \pi^b(p^j; p, w)}{\partial p} = -A_b(p^j; p, w).$$

Now, π^b is the upper envelope of a set of linear (and hence convex) functions of (p^j, p, w) , one function for each choice of (L_b, A_b) . Therefore π^b is a convex function.

Thus, the left side of the envelope equation is increasing in p . This means the right side of the equation is also increasing. Therefore, the apple demand is decreasing in p . In other words, if p goes down, then the beverage firm reacts by increasing A_b .

- (v) What effect on prices and quantities would a lump-sum transfer from the beverage company to the farm have?

Answer. No effect. Adding constants the firms’ objective functions would not affect the firms’ choices. The firms’ profits would change, but since each household owns the same fraction of shares in each firm, the profit changes would exactly cancel out leaving the households’ budget constraints unchanged.

- (vi) * Provide a counter-example to the following false conjecture: $(B(\mathbb{N}, [0, 1]), d_\infty)$ is a compact metric space, where $B(\mathbb{N}, [0, 1])$ is the set of bounded functions from the natural numbers to $[0, 1]$, and $d_\infty(f, g) = \sup_{n \in \mathbb{N}} |f(n) - g(n)|$.

Answer. Consider the sequence $f_n(x) = I(n = x)$, where I is the indicator function. Then $d_\infty(f_n, f_m) = 1$ for all $n \neq m$. Therefore, f_n does not have a convergent subsequence. We conclude that $(B(\mathbb{N}, [0, 1]), d_\infty)$ is not a compact metric space.

(vii) * Prove the following generalisation of Cantor's intersection theorem:

Let (X, d) be a complete metric space. Define the diameter of a set $A \subseteq X$ as $\text{diam}(A) = \sup_{a, b \in A} d(a, b)$. Let $A_n \subseteq X$ be a sequence of non-empty closed sets. If $A_{n+1} \subseteq A_n$ and $\text{diam}(A_n) \rightarrow 0$ then $\bigcap_{n=1}^{\infty} A_n$ contains a single point.

This theorem is adapted from Border and Aliprantis (2005), Section 3.2.

Answer. Let $A^* = \bigcap_{n=1}^{\infty} A_n$.

Uniqueness: Suppose $a^*, a^{**} \in A$, where $a^* \neq a^{**}$. Let $r = d(a^*, a^{**}) > 0$. Then $\text{diam}(A_n) \geq r > 0$ for all n . This contradicts $\text{diam}(A_n) \rightarrow 0$.

Existence: Since each set A_n is non-empty, there exists some point $a_n \in A_n$. We now show that a_n is a Cauchy sequence. Pick any radius $r > 0$. Since $\text{diam}(A_n) \rightarrow 0$, it follows that there exists some N such that $\text{diam}(A_N) < r$. By construction $a_n \in A_N$ for all $n > N$. It follows that $d(a_n, a_m) \leq \text{diam}(A_N) < r$ for all $n, m > N$. So a_n is a Cauchy sequence. Since (X, d) is a complete metric space, a_n is convergent; call the limit a^* . Since each set A_n is closed, $a^* \in A_n$ for all n . Therefore $a^* \in A$.

33: Micro 1, May 2019

A cosmetics firm makes perfume using consulting services. Consulting services are produced from specialised labour and lab materials. There are two consultants, who each owns and supplies labour exclusively to his own consulting firm. The consultants are endowed with the same amount lab materials, which they can sell to any firm. The old consultant has double the human capital of the young consultant. The two consultants own an equal share in the cosmetics firm, and consume perfume only (but not leisure or lab materials).

- (i) Formulate a competitive model of the perfume, consulting, labour, and lab material markets. Hint: you might find it easier to model labour markets as rental markets for human capital.

Comment. One tricky part of this question is that it says that only the young firm can buy young labour (and similarly for the old firm). This idea can be accommodated in general equilibrium theory by assuming that young and old labour are separate goods with separate market clearing conditions and separate prices. A common mistake was to assume the two types of labour traded at the same wage (even if the student wrote two separate market clearing conditions).

In lectures in tutorials, we emphasise a useful double-check: the number of prices must equal the number of market clearing conditions. However, this is not a proof! Patching up a misformulated model by inventing a new price or a new market clearing condition does not fix the problem! The check list is to help you find mistakes, not hide them.

Another common mistake was in specifying the quantities endowed, consumed and sold. If all three variables are included in the model, then they must be connected via a constraint, i.e. consumption plus sales equals endowment. Perhaps a tidier alternative would be to only write down endowments and sales, and to write consumption as a function of the other two.

Answer.

Households. There are two households, $h \in \{y, o\}$. Household h is endowed with m lab materials, k_h human capital, which it sells (or rents) at prices p_m and r_h respectively. The household receives the profit from its firm, π_h , half the profits of the cosmetic firm π_c , and buys perfume c_h at price p_c , which gives utility $u(p_c)$. The household h 's utility maximisation problem is

$$\begin{aligned} \max_{c_h} & u(c_h) \\ \text{s.t.} & p_c c_h = p_m m + r_h k_h + \pi_h + \frac{\pi_c}{2}. \end{aligned}$$

Consultancies. Each consultancy $h \in \{y, o\}$ buys M_h lab materials and K_h human capital to produce $s_h = f(M_h, K_h)$ units of consulting services which it sells at a price of p_s . Its profit function is

$$\pi_h(p_s; p_m, r_h) = \max_{M_h, K_h} p_s f(M_h, K_h) - p_m M_h - r_h K_h.$$

Cosmetics firm. The cosmetics firm chooses how much consulting services S to use, which leads to $C = g(S)$ units of perfume. Its profit function is

$$\pi_c(p_c; p_s) = \max_S p_c g(S) - p_s S.$$

Equilibrium. Prices $(p_m, r_y, r_o, p_s, p_c)$ and quantities $(c_h, M_h, K_h, S_h, S, C)$ constitute an equilibrium if the quantity choices solve the respective optimisation problems above, and markets clear:

$$\begin{aligned} 2m &= M_y + M_o \\ k_y &= K_y \\ k_o &= K_o \\ S_y + S_o &= S \\ C &= c_y + c_o. \end{aligned}$$

- (ii) Suppose that at some non-equilibrium price vector, all markets clear except the labour markets. Does this mean that one of the labour markets has excess supply?

Comment. A common mistake was to declare Walras' law inapplicable because it supposedly only applies to equilibrium price vectors. Of course, Walras' law applies to all price vectors.

Answer. By Walras' law, if markets do not clear, then at least one market has excess supply. Since only the labour markets do not clear, by process of elimination, one of the labour markets has excess supply.

- (iii) Prove that if the price of lab materials increases, then the old consulting firm responds by purchasing fewer lab materials.

Comment. A common mistake was to either claim that the first derivative of the profit function is positive, or that the second derivative is positive. Similarly, when comparing two sides of an equation, it is important to specify which equation you are talking about it.

Some students attempted to apply the first-order condition for the quantity of lab materials. This approach is very difficult to execute correctly and we did not cover it in lectures. (It would be necessary to use the implicit function theorem on the system of equations consisting of both first-order conditions). Many students implicitly assumed that labour demand does not change when the materials price increases.

Answer. By the envelope theorem,

$$\frac{\pi_o(p_s; p_m, r_o)}{p_m} = -M_o(p_s; p_m, r_o).$$

Now, π_o is the upper envelope of a set of linear functions of (p_s, p_m, r) , with one function for each choice of (M_o, K_o) . Therefore, π_o is a convex function. So the left side of the equation above is increasing in p_m . Hence, the right side of the equation is also increasing in p_m . We conclude that the old consultancy's factor demand for lab materials, $M_o(p_s; p_m, r)$ is decreasing in the materials price p_m .

- (iv) Write down the value function for the old consulting firm after it has chosen the labour demand but before it has chosen the demand for lab materials. Write down a Bellman equation linking this to the profit function.

Comment. Many students were familiar with applying dynamic programming to decompose profits into revenue and cost. But this is not what the question was about.

Answer. The relevant value function is

$$V_o(K_o, p_s, p_m) = \max_{M_o} p_s f(M_o, K_o) - p_m M_o.$$

The Bellman equation is

$$\pi_o(p_s; p_m, r) = \max_{K_o} V(k_o, p_s, p_m) - r_o K_o.$$

- (v) Prove that if the consulting production function has constant returns to scale, then the old consultant uses more lab materials than the young consultant. Hint: differentiating $f(tx) = tf(x)$ with respect to x_1 gives $tf_1(tx) = tf_1(x)$.

Comment. Most students started on the wrong track.

Answer. By the market clearing conditions, $K_o = 2K_y = 2k_y$. The first-order conditions for M_h (where $h \in \{y, o\}$) can be arranged as

$$\frac{\partial f(M_h, K_h)}{\partial M} = \frac{p_m}{p_s}.$$

Since the right side is the same for both young and old, we conclude

$$\frac{\partial f(M_y, K_y)}{\partial M} = \frac{\partial f(M_o, K_o)}{\partial M}.$$

Now, since $K_o = 2K_y$ and the constant returns to scale hint, the right side equals

$$\frac{\partial f(M_o, K_o)}{\partial M} = \frac{\partial f(M_o, 2K_y)}{\partial M} = \frac{\partial f(\frac{1}{2}M_o, K_y)}{\partial M}.$$

We conclude that

$$\frac{\partial f(M_y, K_y)}{\partial M} = \frac{\partial f(\frac{1}{2}M_o, K_y)}{\partial M}$$

and hence that $M_y = \frac{1}{2}M_o$.

- (vi) The government would like to increase the amount of perfume production. Either devise a lump-sum transfer scheme that would increase perfume production, or prove that this is impossible.

Answer. This is impossible. If this were true, we could establish that the untaxed equilibrium is inefficient, in violation of the first welfare theorem. Specifically, the taxed allocation could be amended by splitting the surplus perfume equally among the households. Since household utility only depends on perfume consumption,

both would be strictly better off under the amended allocation compared to the untaxed allocation.

Another possible approach is to study the firms' optimisation problems and establish that only one allocation of lab materials can arise in equilibrium.

- (vii) * Consider the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$, the metric space $(X, d) = (C([0, 1]), d_\infty)$, and the set $A = \{f \in X : f(0) \geq 0\}$, where

$$C([0, 1]) = \{f : [0, 1] \rightarrow \mathbb{R}, f \text{ is continuous}\}$$
$$d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|.$$

Is f in the interior of A ?

Answer. No, f is on the boundary of A , and hence not in the interior.

First, $f \in A$, since $f(0) = 0$. Therefore, the trivial sequence $a_n = f$ lies inside A and converges to f .

Second, the sequence $b_n(x) = f(x) - 1/n$ lies outside of A (since $b_n(0) < 0$) and converges to f .

We conclude that f lies in the boundary of A .

34: AME, May 2019

Part A.

Households are endowed with time and vegetables, which they can consume or sell to a restaurant. The restaurant produces meals, which households can also consume.

- (i) Formulate a competitive model of the labour, vegetable and meal markets.

Answer.

Households. There are n identical households. Each household is endowed with e_h hours and e_v vegetables, of which they consume x_h and x_v at prices w and p_v respectively. They receive dividends of π/n . They may also purchase x_m meals at price p_m . Their utility from these quantities is $u(x_h, x_v, x_m)$. The utility maximisation problem is

$$\begin{aligned} \max_{x_h, x_v, x_m} & u(x_h, x_v, x_m) \\ \text{s.t.} & wx_h + p_v x_v + p_m x_m = we_h + p_v e_v + \pi/n. \end{aligned}$$

Restaurant. The restaurant hires H hours of workers and buys V vegetables to produce $M = f(H, V)$ meals. Its profit maximisation problem is

$$\pi(w, p_v, p_m) = \max_{H, V} p_m f(H, V) - wH - p_v V.$$

Equilibrium. The prices (w, p_v, p_m) and quantities (x_h, x_v, x_m, H, V, M) constitute an equilibrium if the quantity choices are optimal in the problems above, and all markets clear:

$$\begin{aligned} nx_h + H &= ne_h \\ nx_v + V &= ne_v \\ nx_m &= M. \end{aligned}$$

- (ii) Suppose the manager of the restaurant has already decided how many vegetables to buy, but the chef still needs to decide how many cooks to hire. Write down the chef's value function. Write down a Bellman equation that connects the chef's value function to the restaurant's profit function.

Comment. Few students answered this part correctly. A common mistake was to double count the revenue, once in the chef's value function and once in the Bellman equation. The latter makes no sense, because the revenue is only determined after the hours are chosen.

Answer. The chef's value function is

$$Z(p_m, w; V) = \max_H p_m f(H, V) - wH.$$

The Bellman equation is

$$\pi(w, p_v, p_m) = \max_V Z(p_m, w, V) - p_v V.$$

(iii) Prove that the chef's value function is convex in prices.

Answer. Z is the upper envelope of a set of linear functions of (p_m, w) , with one function for each choice of H . Therefore Z is a convex function.

(iv) Prove that the chef's demand for cooks is decreasing in the wage of cooks.

Answer. By the envelope theorem,

$$\frac{\partial}{\partial w} Z(p_m, w; V) = -H(p_m, w; V).$$

In the previous part, we established that Z is a convex function, so the left side is increasing in w . Therefore the right side is increasing in w . We conclude that the chef's demand for cooks, $H(p_m, w; V)$, is decreasing in the wage w .

Part B.

(i) Find a metric space (X, d) such that $x_n = \frac{1}{n}$ is not a Cauchy sequence.

Answer. Let $X = \mathbb{R}$ and $d(x, y) = I(x \neq y)$ be the discrete metric. Then $d(x_n, x_m) = 1$ for all $n \neq m$. So x_n is not a Cauchy sequence.

(ii) Find a counterexample to the following false statement: Let (X, d) be a metric space. If A is closed and bounded inside (X, d) , then A is compact.

Answer. Let $A = X = (0, 1)$ and $d = d_2$. A is closed inside (X, d) , because the whole space is closed. A is bounded inside (X, d) , because A is an open ball, e.g. $A = N_1(0.5)$. A is not compact, because the sequence $x_n = \frac{1}{n+2} \in A$ does not have a convergent subsequence.

(iii) Prove that the optimisation problem

$$\max_{x \in [100, 101], y \in \{0, 1\}} (x + y) \sin x$$

has an optimal solution (x^*, y^*) .

Comment. Several students correctly observed that $[100, 101]$ and $\{0, 1\}$ are compact subsets of the real line, but did not then apply this information correctly. The domain of the function is the real plane, not the real line. One way to bridge this gap is to deduce that the choice set $A = [100, 101] \times \{0, 1\}$ is compact. Another way is to split the optimisation problem into two pieces with dynamic programming:

$$\max_{y \in \{0, 1\}} g(y)$$

where

$$g(y) = \max_{x \in [100, 101]} (x + y) \sin x.$$

The extreme value theorem applies to establish that $g(y)$ (and hence an optimal choice of x) exists for all y , and there is an optimal y^* because the finite menu $\{0, 1\}$.

Answer. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = (x + y) \sin x$. Notice that f is continuous. Therefore, the same function f restricted to the domain $A = [100, 101] \times \{0, 1\}$ is also continuous. Since A is a closed and bounded subset of Euclidean space, it is compact (by the Bolzano-Weierstrass theorem.) Therefore, by the extreme value theorem, f has a maximum on A .

- (iv) Let (X, d) be a metric space. Let \mathcal{U} be a set of open sets. Prove or disprove that the union of these sets, $A = \bigcup \mathcal{U}$, is an open set. Note: \mathcal{U} might be an infinite set.

Comment. A common mistake was to start with a particular set $U \in \mathcal{U}$, and then a point $a \in U$. Instead, it is necessary to pick an arbitrary point $a \in A$.

Answer. Pick any $a \in A$. By the construction of A , there must be some $U \in \mathcal{U}$ such that $a \in U$. Since U is open, there is some open ball $N_r(a)$ that is contained in U . Therefore $N_r(a) \subseteq U \subseteq A$, as required.

- (v) Consider the metric spaces (X, d) and (X, d') where $d'(x, y) = \min\{1, d(x, y)\}$. Prove that if (X, d) is complete, then (X, d') is complete.

Comment. A common mistake was to start with an arbitrary Cauchy sequence in (X, d) . But to prove that (X, d') is complete, it is necessary to prove that all Cauchy sequences in (X, d') are convergent.

Answer. Let x_n be a Cauchy sequence in (X, d') . This means that for all $r > 0$, there exists some $N'(r)$ such that

$$d'(x_n, x_m) < r \text{ for all } n, m > N'(r).$$

This implies that x_n is a Cauchy sequence in (X, d) , by setting $N(r) = N'(\min\{1, r\})$. Since (X, d) is complete, it follows that x_n converges to some x^* inside (X, d) . So $d(x_n, x^*) \rightarrow 0$. Since $d'(x_n, x^*) \leq d(x_n, x^*)$, it follows that x_n converges to x^* inside (X, d') .

- (vi) Consider the metric spaces (X, d_X) , (Y, d_Y) and $(CB(X, Y), d_\infty)$, where

$$CB(X, Y) = \{f : X \rightarrow Y, f \text{ is continuous and bounded}\}$$

$$\text{and } d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Prove that if $x_n \in X$ and $f_n \in CB(X, Y)$ are convergent with $x_n \rightarrow x^*$ and $f_n \rightarrow f^*$, then $y_n = f_n(x_n)$ is convergent with $y_n \rightarrow y^* = f^*(x^*)$.

Answer. Pick any $r > 0$. We would like to find N such that

$$d_Y(y_n, y^*) < r \text{ for all } n \geq N.$$

Since $f_n \rightarrow f^*$, we can pick N_1 such that

$$d_\infty(f_n, f^*) < \frac{r}{2} \text{ for all } n \geq N_1.$$

By a theorem in the notes, f^* being continuous at x^* implies that there exists an open ball $N_s(x^*)$ such that $f(N_s(x^*)) \subseteq N_{r/2}(y^*)$. Pick N_2 such that $d(x_n, x^*) < s$ for all $n \geq N_2$.

Putting the two together with the triangle inequality, we conclude that for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} d_Y(y_n, y^*) &\leq d_Y(f_n(x_n), f_N(x_n)) + d_Y(f_N(x_n), y^*) \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

- (vii) Let (X, d) be a compact metric space. Consider the metric space $(C(X), d_\infty)$ of continuous functions $C(X) = \{f : X \rightarrow \mathbb{R}, f \text{ is continuous}\}$ and

$$d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Consider the function $T : C(X) \rightarrow X$ defined by $T(f) = \max_{x \in X} f(x)$. Prove that T is well-defined.

Comment. Many students forgot the definition of “well-defined” – it involves both existence and uniqueness.

Answer. Every function f in the domain of T is continuous and has a compact domain X , so the extreme value theorem implies that f has a maximum value. Therefore, $T(f)$ exists. Maximum values are unique, so $T(f)$ is unique.

- (viii) Let (X, d) be a complete metric space, and let f_n be a sequence of contractions on (X, d) of degree a . Prove that there exists a unique solution x_n^* to the system of equations $x_n = f_n(x_{n+1})$.

Comment. There was a typo in the exam – it read $f(x_{n+1})$ instead of $f_n(x_{n+1})$, although the mistaken notation is actually a sloppy way of writing the same thing. Some students asked about this, but there was some miscommunication and I was unable to see the mistake until afterwards. My apologies.

Answer. Recall that $(\ell_\infty(X), d_\infty)$ is a complete metric space since (X, d) is complete. Construct the function $g : \ell_\infty(X) \rightarrow \ell_\infty(X)$ as $g(\{x_n\}) = \{f_n(x_{n+1})\}$. Now,

$$\begin{aligned} d_\infty(g(\{x_n\}), g(\{y_n\})) &= \sup_n d(f_n(x_{n+1}), f_n(y_{n+1})) \\ &\leq \sup_n ad(x_{n+1}, y_{n+1}) \\ &= a \sup_n d(x_{n+1}, y_{n+1}) \\ &\leq ad_\infty(\{x_n\}, \{y_n\}). \end{aligned}$$

So g is a contraction of degree a on a complete metric space. Therefore, by Banach’s fixed point theorem, g has a unique fixed point x^* . Now $x^* = g(x^*)$ if and only if $x_n^* = f_n(x_{n+1}^*)$ for all n . So x^* is the (only) solution to the system of equations.

35: AME, December 2019

Part A

During the early industrial revolution, trains carried grain from the American midwest to the east coast, and clothing from the east coast to the midwest. Train transport capacity was produced by labour. Capacity was bi-directional, in the sense that the total capacity required was the maximum of the east-bound and west-bound capacity used. Train companies were vertically integrated with retail, i.e. train companies bought and sold grain and clothes. Farms made grain from labour, and factories made clothes from labour. Households in both locations supplied labour and consumed grain and clothes. Train transport could utilise labour from either location, whereas farms only used midwest labour, and factories only used east coast labour. All households had the same labour endowment, and owned the same shares in all firms.

- (i) Formulate a competitive model of the labour, grain and clothing markets in both locations.

Comment. Common mistakes included:

- assuming prices on the east and west are equal,
- allowing prices on the east and west to differ, but then having country-wide market clearing conditions,
- assuming that the number of boxes of clothes and grain transported are equal,
- formulating the train firm's problem with choices that have either no costs or no benefits, and
- having a single household that trades on both in the west and east.

Answer. Households. Let $\ell \in \{E, W\}$ denote location – east or west. For simplicity, assume there is one household in each location. The household located at ℓ chooses the amount of labour h_ℓ to supply, and the amount of grain g_ℓ and clothing c_ℓ to consume at prices w_ℓ , p_ℓ , and q_ℓ respectively. For convenience in modelling freight, we measure grain and clothing quantities in “boxes”. This gives the household utility $u(1 - h_\ell, g_\ell, c_\ell)$. The household receives dividends $\pi_t/2$, $\pi_g/2$, $\pi_c/2$ from the train firm, farm and factory, respectively. The household's utility maximisation problem is

$$\begin{aligned} \max_{h_\ell, g_\ell, c_\ell} & u(1 - h_\ell, g_\ell, c_\ell) \\ \text{s.t.} & p_\ell g_\ell + q_\ell c_\ell = w_\ell h_\ell + \pi_t/2 + \pi_g/2 + \pi_c/2. \end{aligned}$$

Farm. The farm hires west labour H_g to make $G = f_g(H_g)$ boxes of grain, which are sold in the west. Its profit function is

$$\pi_g(p_W; w_W) = \max_{H_g} p_W f(H_g) - w_W H_g.$$

Factory. The factory hires east labour H_c to make $C = f_c(H_c)$ boxes of clothing, which are sold in the east. Its profit function is

$$\pi_c(q_E; w_E) = \max_{H_c} q_E f(H_c) - w_E H_c.$$

Train company. The train company buys grain from the west G_t which it sells in the east, and clothes in the east C_t , which it sells in the west. To do this, it requires $\max\{G_t, C_t\}$ capacity. It can produce a capacity of $f_t(H_{tE} + H_{tW})$ boxes with labour from both locations H_{tE} and H_{tW} – they are perfect substitutes. The profit function is

$$\begin{aligned} \pi_t(p_W, q_E; p_E, q_W, w_W, w_E) &= \max_{G_t, C_t, H_{tW}, H_{tE}} (p_E - p_W)G_t + (q_W - q_E)C_t - w_W H_{tW} - w_E H_{tE} \\ \text{s.t. } & f_t(H_{tE} + H_{tW}) = \max\{G_t, C_t\}. \end{aligned}$$

Equilibrium. Prices $(w_W, w_E, p_W, p_E, q_W, q_E)$ and quantities

$$(h_E, g_E, c_E, h_W, g_W, c_W, H_g, H_c, G_t, C_t, H_{tW}, H_{tE})$$

constitute an equilibrium if the quantities solve respective optimisation problems above, and all markets clear:

$$\begin{aligned} h_E &= H_c + H_{tE} \\ h_W &= H_g + H_{tW} \\ c_E + C_t &= f_c(H_c) \\ c_W &= C_t \\ g_W + G_t &= f_g(H_g) \\ g_E &= G_t. \end{aligned}$$

(ii) Prove that the train firm's profit function is convex.

Answer. The firm's objective is linear prices. (Notice that prices don't enter the constraint.) So the profit function is the upper envelope of linear functions, and is therefore convex.

(iii) Prove that the train firm reacts to an increase in the midwest price of grain by trading less grain.

Answer. By the envelope theorem,

$$\begin{aligned} & \frac{\partial \pi_t(p_W, q_E; p_E, q_W, w_W, w_E)}{\partial p_W} \\ &= \left[\frac{\partial}{\partial p_W} \{(p_E - p_W)G_t + (q_W - q_E)C_t - w_W H_{tW} - w_E H_{tE}\} \right]_{\text{at optimal } (G_t, C_t, h_{tW}, h_{tE})} \\ &= [-G_t]_{\text{at optimal } (G_t, C_t, h_{tW}, h_{tE})} \\ &= -G_t(p_W, q_E; p_E, q_W, w_W, w_E). \end{aligned}$$

Since we established that π is convex, we know the left side, and hence the right side, are increasing in p_W . Therefore, the amount of grain traded, $G_t(p_W, q_E; w_W, w_E)$, is decreasing in the midwest price of grain p_W .

- (iv) The train companies (“robber barons”) often purchased upstream suppliers. Write down the profit function of the train firm after it purchased the other firms as a Bellman equation.

Comment. A common mistake was to write a Bellman equation using the cost minimisation problem – this misses the point of the question.

Answer. The integrated firm’s profits are simply the sum of the profits of its divisions.

$$\pi_T(w_W, w_E, p_W, p_E, q_W, q_E) = \pi_g(p_W; w_W) + \pi_c(q_E; w_E) + \pi_t(p_W, q_E; p_E, q_W, w_W, w_E).$$

Part B

- (i) Consider the metric space (X, d_1) where $X = [0, 2] \times \{0, 1\}$ and $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$. What is the interior of the set $A = [0, 1] \times \{0\}$ inside (X, d_1) ?

Comment. Although it seems most students knew what openness means, most students struggled to apply it to this specific set.

Answer. The interior is $B = [0, 1) \times \{0\}$. First, to see that B is an open set, pick any point $b \in B$. Let $r = \min\{b_1, 1 - b_1\}$. Then the open ball $B_r(b) \subseteq B$.

Second, the only point from A that is missing from B is $(1, 0)$, i.e. $A \setminus B = \{(1, 0)\}$. Now $(1, 0)$ is not in the interior of A , because it is on the boundary – the sequence $x_n = (1 + 1/n, 0)$ converges to $(1, 0)$.

Therefore, B contains all of the interior points of A .

- (ii) Suppose A is a subset inside the metric space (X, d) . Prove that if A is both closed and open, then the boundary of A is empty, i.e. $\partial A = \emptyset$.

Comment. Many students did well in this question. However, many students strayed from the definitions, possibly because they were uncomfortable writing with mathematical quantifiers (“there exists” and “for all”).

Answer. We know that A is closed implies $\partial A \subset A$. We also know that A is open implies $\partial A \cap A = \emptyset$. In other words, if $x \in \partial A$ then $x \in A$ and $x \notin A$. Therefore A must be empty.

- (iii) Let (X, d) be a metric space. Prove that if $A \subseteq X$ and (A, d) is a complete metric space, then A is a closed set inside (X, d) .

Comment. This question was also generally well done. Some students got caught up in sequences and sequences of sequences. They meant the right thing, but had difficulty in handling the sequences correctly.

Answer. Suppose $a_n \in A$ is a convergent sequence with limit x^* . We want to prove that $x^* \in A$. Since a_n is convergent, a_n is a Cauchy sequence inside (X, d) , and hence inside (A, d) also (since distances are measured the same way). Since a_n is a Cauchy sequence inside a complete metric space (A, d) , it is convergent, with some limit $a^* \in A$. So $a_n \rightarrow a^*$ inside the metric space (X, d) . Since a_n converges to both a^* and x^* , and sequences converge to at most one point, we conclude that $a^* = x^*$.

(iv) Give an example of a complete and bounded metric space that is not compact.

Comment. Most students wisely went straight for the discrete metric, but struggled to write coherently. A common mistake was to not think carefully about the point set.

Answer. Consider the space $([0, 1], d)$, where d is the discrete metric. This space is complete and bounded, but not compact. All discrete spaces are:

- complete, because all Cauchy sequences eventually become trivial – the same point repeated over, and
- bounded, because distances never exceed 1.

This space is not compact, because the sequence $x_n = 1/n$ does not have a convergent subsequence – $d(x_n, x_m) = 1$ for all $n \neq m$.

Another answer. $(B([0, 1], [0, 1]), d_\infty)$ is complete and bounded, but not compact. We proved that this space is complete in class. It is bounded because $B([0, 1], [0, 1]) = B_1(f^*)$ where $f^*(x) = 0$. It is not compact, because the sequence

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

does not have a convergent subsequence, since $d_\infty(f_n, f_m) = 1$ for all $n \neq m$.

(v) Prove that if X is a finite set, then the metric space (X, d) is compact.

Comment. Common mistakes on this question included:

- misunderstanding sequences, e.g. that they are infinite, and
- trying to apply the Bolzano-Weierstrass theorem – which is only applicable in (\mathbb{R}^n, d_2) .

Answer. Consider any sequence $x_n \in X$. We would like to find a convergent subsequence. Since X is finite, there must be at least one point, x^* that appears infinitely often in x_n . Therefore, $y_n = x^*$ is a subsequence of x_n that converges to x^* .

(vi) Consider a public good contributions game in which player 1 donates x and player 2 donates y . Suppose player 1 wants to donate $x = f(y)$ and player 2 wants to donate $y = g(x)$, where f and g are decreasing differentiable functions with $f'(y) > -a$ for all x and $g'(x) > -a$ for all y , and a is some number in $(0, 1)$. Prove that there is a unique equilibrium, i.e. (x^*, y^*) such that $x^* = f(y^*)$ and $y^* = g(x^*)$.

Comment. Common mistakes included:

- not thinking about function composition, i.e. $h(x) = f(g(x))$,
- incorrectly deducing that the functions are concave or convex, and
- incorrectly deducing that the derivatives equal $-a$.

Answer. Let $h(x) = f(g(x))$. Then h is increasing and $h'(x) = f'(g(x))g'(x) < a^2 < 1$. So h is a contraction of degree a^2 . To see this, notice that if $x_2 > x_1$, then

$$d_2(h(x_1), h(x_2)) = h(x_2) - h(x_1) = \int_{x_1}^{x_2} h'(x) dx \leq \int_{x_1}^{x_2} a^2 dx = (x_2 - x_1)a^2 = a^2 d_2(x_1, x_2).$$

Since $d_2(x_1, x_2) = d_2(x_2, x_1)$, this inequality holds for $x_2 \leq x_1$ as well.

Since h is a contraction on a complete metric space (\mathbb{R}, d_2) , Banach's fixed point theorem implies that there is a unique $x^* \in \mathbb{R}$ such that $x^* = h(x^*)$.

Let $y^* = g(x^*)$. Now, $x^* = h(x^*) = f(g(x^*)) = f(y^*)$. We conclude that $(x^*, y^*) = (f(y^*), g(x^*))$, so (x^*, y^*) is an equilibrium.

Finally, suppose (\hat{x}, \hat{y}) is an equilibrium, i.e. that $(\hat{x}, \hat{y}) = (f(\hat{y}), g(\hat{x}))$. Then $\hat{x} = f(\hat{y}) = f(g(\hat{x})) = h(\hat{x})$ is a fixed point of h . Since h has only one fixed point, $\hat{x} = x^*$ and $\hat{y} = g(x^*) = y^*$. So (x^*, y^*) is the only equilibrium.

- (vii) Consider a social planner who would like to distribute an endowment $e > 0$ among a society of n individuals to maximise welfare $W(x) = \sum_{i=1}^n u_i(x_i)$, where each individual's utility function $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous. Prove that there is a solution to the social planner's problem,

$$\begin{aligned} & \max_{x \in \mathbb{R}_+^n} W(x) \\ & \text{s.t. } \sum_{i=1}^n x_i = e. \end{aligned}$$

Comment. Common mistakes included:

- incorrectly stating that (\mathbb{R}, d_2) is compact,
- not specifying the menu,
- arguing that any subset of a compact set is compact,
- applying a fixed point theorem, even though this is an optimization problem.

Answer. First, the welfare function $W : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is continuous, because the sum of continuous functions is continuous.

Second, consider the menu $A = \{x \in \mathbb{R}_+^n : \sum_i x_i = e\}$. It is closed, because $A = f^{-1}(\{e\})$ where $f(x) = \sum_i x_i$ is a continuous function and $\{e\}$ is a closed set. It is bounded because $A \subseteq B_{ne}(0)$. So A is a compact set inside (\mathbb{R}^n, d_2) by the Bolzano-Weierstrass Theorem. So (A, d_2) is a compact metric space.

Since W is continuous on the domain (\mathbb{R}_+^n, d_2) , it is also continuous on the domain (A, d_2) . Since A is compact, the Extreme Value Theorem implies that the problem

$$\max_{x \in A} W(x)$$

has a solution.

- (viii) Suppose a boiler's energy efficiency degrades over time, but can be restored. Let x be the boiler's efficiency – measured in the amount of energy needed to heat a building for one day, p be the price of energy, $r(x, x')$ be the repair cost to restore x to x' (which might be greater than zero even if $x' > x$, due to degradation). Assume that r is continuous. Money is discounted at rate β . The value of boilers $\pi(x)$ solves the Bellman equation

$$\pi(x) = \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi(x').$$

Recall: $\inf A$ is the largest number that is weakly smaller than everything in A , e.g. $\inf(0, 1] = 0$.

- (a) Reformulate the Bellman equation as a fixed point problem.

Comment. Common mistakes included:

- putting the Bellman operator on both sides of the equation, and
- not specifying the domain and co-domain of the Bellman operator.

Answer. Let $F : B[0, 1] \rightarrow B[0, 1]$ be the function defined by

$$F(\pi)(x) = \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi(x').$$

Then the Bellman equation can be reformulated as $\pi = F(\pi)$.

- (b) Assume that the function in the previous part is a contraction. Suppose that $r(x, x')$ is concave in x . Prove that the solution to the Bellman equation, π , is concave. *Hint: this proof has several steps. As always, you can get credit for any of the steps.*

Comment. A common mistake was to attempt to prove that the Bellman operator is concave.

These sample solutions include a proof that the Bellman operator is a contraction, even though the question says that you should assume this is true. Note that you can get credit for giving a more complete answer, and you can also get credit for giving partial answers. (Recall that the amount of credit is determined by the nature of the snippets of logic that you write, not the proportion of the question that you answered.)

Answer. F is a contraction of degree β on $(B[0, 1], d_\infty)$. Pick any $\pi, \pi' \in B[0, 1]$. We would like to prove that $d_\infty(F(\pi), F(\pi')) \leq \beta d_\infty(\pi, \pi')$. Now, for all $x \in [0, 1]$ we have

$$\begin{aligned} F(\pi)(x) &= \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi(x') \\ &= \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi'(x') - \beta\pi'(x') + \beta\pi(x') \\ &\leq \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi'(x') + \sup_{x' \in [0,1]} -\beta\pi'(x') + \beta\pi(x') \\ &\leq \inf_{x' \in [0,1]} px + r(x, x') + \beta\pi'(x') + \sup_{x' \in [0,1]} |-\beta\pi'(x') + \beta\pi(x')| \\ &= F(\pi')(x, p) + \beta d_\infty(\pi, \pi'). \end{aligned}$$

Rearranging, we find for all $x \in [0, 1]$ that

$$F(\pi)(x) - F(\pi')(x) \leq \beta d_\infty(\pi, \pi').$$

By reversing the role of π and π' we also find

$$F(\pi')(x) - F(\pi)(x) \leq \beta d_\infty(\pi, \pi').$$

Therefore,

$$|F(\pi')(x) - F(\pi)(x)| \leq \beta d_\infty(\pi, \pi').$$

Taking a supremum, we conclude that

$$d_\infty(F(\pi), F(\pi')) \leq \beta d_\infty(\pi, \pi').$$

F has a unique fixed point π^* on $(B[0, 1], d_\infty)$. In class, we proved that $(B[0, 1], d_\infty)$ is a complete metric space (when the co-domain – in this case (\mathbb{R}, d_2) – is complete). Since F is a contraction on a complete metric space, Banach's fixed point theorem implies that F has a unique fixed point, which we call π^* .

F is a self-map on the set of bounded concave functions. Let $CCB[0, 1] = \{\pi \in B[0, 1] : \pi \text{ is concave}\}$. Suppose $\pi \in CCB[0, 1]$, i.e. that π is concave. We would like to prove that $F(\pi) \in CCB[0, 1]$, i.e. that $F(\pi)$ is a concave function. Since p, π and r are concave functions of x , the objective

$$x \mapsto px + r(x, x') + \beta\pi(x', \beta p)$$

is concave. Now,

$$F(\pi)(x) = \inf_{x' \in [0, 1]} px + r(x, x') + \beta\pi(x')$$

is the lower envelope of concave functions – one function for each x' . Therefore, $F(\pi)$ is concave.

$(CCB[0, 1], d_\infty)$ is a complete metric space. We will prove that $CCB[0, 1]$ is a closed set inside $(B[0, 1], d_\infty)$. Since the latter is a complete metric space, this implies that $(CCB[0, 1], d_\infty)$ is a complete metric space.

Suppose $\pi_n \in CCB[0, 1]$ is a convergent sequence with the limit $\pi^* \in B[0, 1]$. We would like to prove that $\pi^* \in CCB[0, 1]$, i.e. that π^* is concave. Specifically, pick any $x, x', t \in [0, 1]$. We would like to prove that

$$t\pi^*(x) + (1 - t)\pi^*(x') \leq \pi^*(tx + (1 - t)x').$$

Now,

$$\begin{aligned} & t\pi^*(x) + (1 - t)\pi^*(x') \\ &= t \lim_{n \rightarrow \infty} \pi_n(x) + (1 - t) \lim_{n \rightarrow \infty} \pi_n(x') \quad (\text{since } \pi_n \rightarrow \pi^*) \\ &= \lim_{n \rightarrow \infty} t\pi_n(x) + (1 - t)\pi_n(x') \\ &\leq \lim_{n \rightarrow \infty} \pi_n(tx + (1 - t)x') \quad (\text{since } \pi_n \text{ is concave}) \\ &= \pi^*(tx + (1 - t)x') \quad (\text{since } \pi_n \rightarrow \pi^*), \end{aligned}$$

as required.

π^* **is concave.** Since F is a contraction on the complete metric space $(CCB[0, 1], d_\infty)$, it has a unique fixed point $\pi^{**} \in CCB[0, 1]$. Now, since $\pi^{**} \in B[0, 1]$, it is also a fixed point of F in this bigger space. But we already found that π^* is the unique fixed of F on $(B[0, 1], d_\infty)$. So $\pi^* = \pi^{**}$. We conclude that π^* is concave.

36: Micro 1, December 2019

After the Forth bridge was built in 1889, trade and commuting between Fife and Edinburgh became much easier.

Coal is produced from labour. Garments (clothes) are produced from coal and labour. Assume that both coal and garments can be produced in both places, but that coal is easier to produce in Fife, and garments are easier to produce in Edinburgh. Before the bridge was completed, Edinburgh and Fife were autonomous, i.e. there was no trade between them. Afterwards, workers from both places could commute and work in either place, and coal and garments were traded. Assume that households have discounted utility, with the same per-period utility function. Assume that all households are identical in terms of endowments and preferences apart from (i) their locations, and (ii) that all firms are owned locally. Neither coal nor garments are storable.

Comment. This exam question was hard. The model was complex, requiring at least nine markets, and most of the questions required applying the tools in clever ways. This means that the top students had a chance to shine, but some students in the middle did not have a chance to show what they knew. On the other hand, Part B of the exam was easy, so in most cases this cancelled out any effect on students' grades.

- (i) Formulate a competitive model of the labour, coal and garments markets operating before and after the Forth bridge was completed. *Hint: any correct answer has more than seven markets.*

Comments (by Sean Ferguson). Answers that went wrong tended to struggle with what happens after the bridge opens. Typically they either failed to account for the possibility of trade between Edinburgh and Fife in the second period (simply repeating the first period, with separate markets for each good in each region) or had inconsistencies between the prices and market clearing conditions; for example, having market clearing conditions that implied a unified second-period market in each good, but maintaining per-region prices, leading to more prices than market clearing conditions. Some answers attempted to maintain the separate markets in each region while allowing firms and consumers to choose how much to supply to or demand from each market. This is not inherently wrong (although since the goods are perfect substitutes and can be traded freely, they would have the same price in equilibrium), but it is unnecessarily complicated and lead to mistakes in almost all cases. Other common mistakes included failing to capture the productivity differences described in the question (either with a regional productivity parameter or with different production functions in each region), giving the households separate per-period budget constraints instead of one overall budget constraint (this rules out saving or borrowing by assumption), and failing to include coal used in garment production in the market clearing conditions for coal.

Answer. Households. For simplicity, assume that there are two households, $\{E, F\}$, living in Edinburgh and Fife respectively. There are two time periods $t \in \{1, 2\}$, before and after the bridge respectively. Each household h chooses labour supply ℓ_{ht} , coal consumption c_{ht} and garment consumption g_{ht} in each period to maximise utility

$$u(\ell_{h1}, c_{h1}, g_{h1}) + \beta u(\ell_{h2}, c_{h2}, g_{h2}).$$

Before the bridge opens, these trade at prices w_{h1} , p_{h1} , and q_{h1} respectively. After the bridge opens, these trade at prices w_2 , p_2 , and q_2 respectively. Local firms profits are denoted $\Pi_h = \Pi_h^c + \pi_h^g$, defined below. Household h 's utility maximisation problem is

$$\begin{aligned} & \max_{(\ell_{ht}, c_{ht}, g_{ht})_{t \in \{1,2\}}} u(\ell_{h1}, c_{h1}, g_{h1}) + \beta u(\ell_{h2}, c_{h2}, g_{h2}) \\ \text{s.t. } & p_{h1}c_{h1} + p_2c_{h2} + q_{h1}g_{h1} + q_2g_{h2} = w_{h1}\ell_{h1} + w_2\ell_{h2} + \Pi_h. \end{aligned}$$

Coal firms. There are two coal firms $x \in \{E, F\}$ located in Edinburgh and Fife respectively. Coal firm x chooses labour demand L_{xt}^c in period t , and produces $C_{xt} = f_x^c(L_{xt}^c)$ units of coal. Its profit maximisation problem is

$$\pi_x^c(p_{x1}, p_2; w_{x1}, w_2) = \max_{L_{x1}^c, L_{x2}^c} p_{x1}f_x^c(L_{x1}^c) - w_{x1}L_{x1}^c + p_2f_x^c(L_{x2}^c) - w_2L_{x2}^c.$$

Garment firms. There are two garment firms $x \in \{E, F\}$ located in Edinburgh and Fife respectively. Garment firm x chooses labour demand L_{xt}^g and coal input C_{xt}^g in period t , and produces $G_{xt} = f_x^g(L_{xt}^g, C_{xt}^g)$ garments. Its profit maximisation problem is

$$\begin{aligned} & \pi_x^g(q_{x1}, q_2; w_{x1}, w_2, p_{x1}, p_2) \\ & = \max_{L_{x1}^g, L_{x2}^g, C_{x1}^g, C_{x2}^g} q_{x1}f_x^g(L_{x1}^g, C_{x1}^g) - w_{x1}L_{x1}^g - p_{x1}C_{x1}^g + q_2f_x^g(L_{x2}^g, C_{x2}^g) - w_2L_{x2}^g - p_2C_{x2}^g. \end{aligned}$$

Equilibrium. Prices $(w_{E1}, w_{F1}, w_2, p_{E1}, p_{F1}, p_2, q_{E1}, q_{F1}, q_2)$ and quantities

$$(\ell_{ht}, c_{ht}, g_{ht}, L_{ht}^c, L_{ht}^g, C_{ht}, C_{ht}^g, G_{ht})_{t \in \{1,2\}, h \in \{E, F\}}$$

form an equilibrium if the quantities solve the above optimisation problems, and all markets clear:

$$\begin{aligned} \ell_{E1} &= L_{E1}^c + L_{E1}^g \\ \ell_{F1} &= L_{F1}^c + L_{F1}^g \\ \ell_{E2} + \ell_{F2} &= L_{E2}^c + L_{F2}^c + L_{E2}^g + L_{F2}^g \\ c_{E1} + C_{E1}^g &= C_{E1} \\ c_{F1} + C_{F1}^g &= C_{F1} \\ c_{E2} + c_{F2} + C_{E2}^g + C_{F2}^g &= C_{E2} + C_{F2} \\ g_{E1} &= G_{E1} \\ g_{F1} &= G_{F1} \\ g_{E2} + g_{F2} &= G_{E2} + G_{F2}. \end{aligned}$$

- (ii) Re-formulate the Edinburgh households' utility maximisation problems by burying the post-bridge-opening choices inside a value function.

Comment (by Sean Ferguson). Many answers made some attempt at rewriting the households problem with the expenditure function, which is not what the question was asking for. Answers that went in the right direction without entirely getting there tended to have problems with the budget constraints, in particular dropping the budget constraint from the second-period value function and/or including second-period choice variables in the first-period budget constraint.

Answer.

$$\begin{aligned} & \max_{\ell, c, g, a'} u(\ell, c, g) + \beta V(a') \\ \text{s.t. } & p_{E1}c + q_{E1}g + a' = w_{E1}\ell + \Pi_E \end{aligned}$$

where

$$\begin{aligned} V(a) &= \max_{\ell, c, g} u(\ell, c, g) \\ \text{s.t. } & p_2c + q_2g = w_2\ell + a. \end{aligned}$$

- (iii) Prove that in every equilibrium, Edinburgh households neither save nor borrow (where dividends from profits earned in each period are attributed to that period).

Comment (by Sean Ferguson). Many students wrote that saving and borrowing are impossible because the goods are not storable. This is a fundamental misunderstanding. ‘Saving’ does not require storing goods. It simply means that a household chooses first-period demand with a lower market value than its first-period income, allowing it to choose second-period demand with a higher market value than its second-period income. Essentially, saving households trade first-period goods to borrowing households in return for second-period goods. This does not require storable goods, but it does require different households making different consumption decisions.

Answer. Intuitively, savings are impossible, because the Edinburgh household has no-one to lend to in the first period. This can be shown mathematically as follows. Let π_{E1} be the profits earned by Edinburgh firms in the first period. The net savings in the first period are

$$\begin{aligned} & w_{E1}\ell_{E1} - p_{E1}c_{E1} - q_{E1}g_{E1} + \Pi_{E1} \\ &= w_{E1}\ell_{E1} - p_{E1}c_{E1} - q_{E1}g_{E1} + [p_{E1}C_{E1} - w_{E1}L_{E1}^c] + [q_{E1}G_{E1} - w_{E1}L_{E1}^g - p_{E1}C_{E1}^g] \\ &= -p_{E1}c_{E1} - q_{E1}g_{E1} + [p_{E1}C_{E1}] + [q_{E1}G_{E1} - p_{E1}C_{E1}^g] \\ &= -q_{E1}g_{E1} + q_{E1}G_{E1} \\ &= 0, \end{aligned}$$

where the cancellations from the market clearing conditions in the Edinburgh labour, coal and garment markets, respectively.

- (iv) Suppose that when the bridges open, the real wages in Edinburgh in terms of coal increases, i.e. the ratio of wages divided by the price of coal in Edinburgh increases. Prove that Edinburgh decreases its coal production.

Comment (by Sean Ferguson). Many answers attempted to use the usual Envelope theorem + convexity of the profit function argument that shows production decreases when wages increase. An argument along these lines is possible, but there is a snag – the standard argument is for when wages change holding other prices constant, whereas here both prices may have changed (note that w/p increasing does not, as some answers argued, mean that both w increases and p decreases!). The trick, which the best answers hit on, is to transform the optimisation problem by dividing out the coal price – this does not change the optimal choices, and gives a new value function in which the labour input is multiplied by w/p rather than w , allowing the standard argument to proceed.

Answer. Suppose

$$\frac{w_{1E}}{p_{1E}} < \frac{w_2}{p_2}.$$

We would like to prove that $C_{1E} < C_{2E}$.

The first-order conditions for L_{1E}^c and L_{2E}^c can be written as

$$\begin{aligned} f_L^c(L_{1E}) &= \frac{w_{1E}}{p_{1E}} \\ f_L^c(L_{2E}) &= \frac{w_2}{p_2}. \end{aligned}$$

Comparing the right sides, we assumed that the top numbers (on both sides) are smaller than the bottom numbers. Therefore, comparing the left sides, we find that $f_L^c(L_{1E}) < f_L^c(L_{2E})$. If we assume that f^c is strictly concave, then f_L^c is strictly decreasing and hence $L_{1E} > L_{2E}$. If we assume that f^c is increasing, then we deduce that $C_{1E} = f(L_{1E}) > f(L_{2E}) = C_{2E}$.

- (v) Prove that in every equilibrium, welfare in Edinburgh increases after the bridge opens.

Comment (by Sean Ferguson). This was a challenging question and no answers got it right. Most answers attempted to use the welfare theorems, typically stating that Edinburgh households could not be worse off after the bridge opens because the equilibrium is efficient. Even if both periods are considered as separate equilibria (reasonable given part iii), the fact that both are efficient does not by itself imply that nobody gets worse off, just that someone else (Fife households in this case) would have to get better off.

Answer. By substituting the firms' profit functions into their owners' budget constraints, we can establish that any equilibrium is also an equilibrium in a corresponding pure-exchange economy with home production.

Now consider the pure-exchange economy. Opening the bridge means that households can trade labour, coal and garments in the second period. If the Edinburgh household chooses not to trade, then it would face the same decision problem as the first period, and get the same utility. If it chooses to trade, then this implies it gets higher utility from trade. Therefore, welfare in Edinburgh is (weakly) higher after the bridge opens.

- (vi) Prove that in every equilibrium, after the bridge opens, Edinburgh produces more garments than Fife.

Comment (by Sean Ferguson). All good answers to this question used an approach based on the firms' first-order conditions and the fact that both firms face the same prices after the bridge opens. Most used an assumption that marginal productivities are higher at the same level of inputs to capture 'easier to produce'. It is also possible to use a productivity parameter, which works with either first-order conditions or an Envelope Theorem approach as shown in the sample solutions.

Answer. Suppose that the Edinburgh and Fife garment production functions are of the form

$$z_x f^g(L, C),$$

where the productivity parameter is higher in Edinburgh than Fife, i.e. $z_E > z_F$. Let

$$\pi^g(q, w, p) = \max_{L, C} q f^g(L, C) - wL - pC.$$

Then the Edinburgh and Fife second-period garment profits can be calculated as $\pi^g(z_E q_2, w_2, p_2)$ and $\pi^g(z_F q_2, w_2, p_2)$, respectively. By the envelope theorem,

$$\begin{aligned} \frac{\partial \pi^g(q, w, p)}{\partial q} &= \left[\frac{\partial}{\partial q} \{q f^g(L, C) - wL - pC\} \right]_{L=L(q, w, p), C=C(q, w, p)} \\ &= [f^g(L, C)]_{L=L(q, w, p), C=C(q, w, p)} \\ &= f^g(L(q, w, p), C(q, w, p)) \\ &= G(q, w, p). \end{aligned}$$

Now, π^g is the upper envelope of a collection of linear (and hence convex) functions of (q, w, p) . So π^g is convex. Thus, the left side of the envelope condition is increasing in q . We conclude that the right side – garment output – is increasing in q . Since $z_E > z_F$, we conclude $G(z_E q_2, w_2, p_2) > G(z_F q_2, w_2, p_2)$.

37: AME, May 2020

Part A

A machine learning (ML) firm has to choose between Canada and Germany for its next data centre. The ML firm uses energy and labour to provide ML services. Both energy and labour must be supplied locally (for energy efficiency and security reasons). A car firm uses energy, ML services and labour – all from any location – to make electric cars. Households are endowed with energy and labour which they supply to the market, and they consume cars. There is a single global market for cars and ML, but separate markets in each country for labour and energy.

- (i) Formulate a competitive model of the energy, labour, ML services, and electric car markets in Canada and Germany.

Comment. Common mistakes included:

- Not formulating the ML firm's problem as having a location choice.
- Assuming that all households are located in both Germany and Canada.
- Assuming a single labour market and/or a single energy market.

Answer. Households. The representative household in country $i \in \{C, G\}$ is endowed with energy e_i and labour ℓ_i , which they sell at prices q_i and w_i . They choose how many cars c_i to purchase, at the price p . This gives them a utility $u(c_i)$. They also receive an equal share $(\pi_{ML} + \pi_c)/2$ of the firms' profits. Their utility maximisation problem is

$$\begin{aligned} \max_{c_i} u(c_i) \\ \text{s.t. } pc_i = w_i \ell_i + q_i e_i + (\pi^{ML} + \pi^c)/2. \end{aligned}$$

ML firm. The ML firm chooses which country I_{ML} to operate in, how much energy E_{ML} to use and how much labour L_{ML} to hire, and produces $S_{ML} = f_{ML}(E_{ML}, L_{ML})$ units of ML services. It sells these services at price r . Its profit function is

$$\pi_{ML}(r; w_C, w_G, q_C, q_G) = \max_{I_{ML}, E_{ML}, L_{ML}} r f_{ML}(E_{ML}, L_{ML}) - q_{I_{ML}} E_{ML} - w_{I_{ML}} L_{ML}.$$

Car firm. The car firm chooses how much energy (E_c^G, E_c^C) and labour (L_c^G, L_c^C) from each country to use, how much ML services S_c to use. The car firm uses these inputs to produce $C_c = g(E_c^G + E_c^C, L_c^G + L_c^C, S_c)$ cars. Its profits are

$$\begin{aligned} \pi_c(p; w_C, w_G, q_C, q_G, r) \\ = \max_{E_c^G, E_c^C, L_c^G, L_c^C, S_c} pg(E_c^G + E_c^C, L_c^G + L_c^C, S_c) - q_G E_c^G - q_C E_c^C - w_G L_c^G - w_C L_c^C - r S_c. \end{aligned}$$

Equilibrium. The prices $(w_C, w_G, q_C, q_G, r, p)$ and quantities

$$(\ell_C, \ell_G, c_C, c_G, I_{ML}, E_{ML}, L_{ML}, S_{ML}, E_c^G, E_c^C, L_c^G, L_c^C, S_c, C_c)$$

constitute an equilibrium if the quantities solve the optimisation problems above, and all markets clear, i.e.

$$\begin{aligned}\ell_C &= L_{ML}1(I_{ML} = C) + L_c^C \\ \ell_G &= L_{ML}1(I_{ML} = G) + L_c^G \\ e_C &= E_{ML}1(I_{ML} = C) + E_c^C \\ e_G &= E_{ML}1(I_{ML} = G) + E_c^G \\ S_C &= S_{ML} \\ c_C + c_G &= C_c.\end{aligned}$$

- (ii) Suppose the ML firm chooses Canada. Prove that the ML firm reacts to a Canadian wage rise by hiring fewer Canadian workers.

Answer. There are two possibilities: either

- the Canadian wage rise leads the ML firm to (re)locate to Germany – in which case it fires all of its Canadian workers (if any), or
- the wage rise does not affect its country choice.

In the second case, we apply the envelope theorem to calculate the ML firm's marginal profit of Canadian wages:

$$\begin{aligned}\frac{\partial}{\partial w_C} \pi_{ML}(r; w_C, w_G, q_C, q_G) \\ &= \frac{\partial}{\partial w_C} [r f_{ML}(E_{ML}, L_{ML}) - q_C E_{ML} - w_C L_{ML}] \\ &= -L_{ML}(r; w_C, w_G, q_C, q_G).\end{aligned}$$

Now, π_{ML} is the upper envelope of linear functions of prices (one for each choice of (I_{ML}, E_{ML}, L_{ML})). So it is the upper envelope of convex functions, and therefore π_{ML} is a convex function. Therefore, the left side (and hence both sides) of the equation above is increasing in w_C . We conclude that $L_{ML}(r; w_C, w_G, q_C, q_G)$ is decreasing in w_C .

- (iii) Formulate the ML firm's problem as a Bellman equation involving the location choice only.

Comment. Less than half of the students got this right. Many students wrote down a Bellman equation involving a cost function (but not location choice). I suspect the students knew they were on the wrong track, but wanted to write something down.

Answer. First, we construct profit functions based local prices (w, q) :

$$\hat{\pi}_{ML}(r; w, q) = \max_{E_{ML}, L_{ML}} r f_{ML}(E_{ML}, L_{ML}) - q E_{ML} - w L_{ML}.$$

Second, the ML firm's profit function satisfies the following Bellman equation:

$$\pi_{ML}(r; w_C, w_G, q_C, q_G) = \max \{ \hat{\pi}_{ML}(r; w_C, q_C), \hat{\pi}_{ML}(r; w_G, q_G) \}.$$

Part B

- (i) Either give an example of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with the property that $f([0, 1]) = [0, 1]$, or prove that this is impossible.

Answer. $f(x) = \sin \pi x$.

- (ii) Either give an example of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with the property that $f([0, 1]) = [0, 1)$, or prove that this is impossible.

Answer. This is impossible. Pick any continuous function f . In lectures, we proved that if A is a compact set and f is a continuous function then $f(A)$ is compact. Now, $[0, 1]$ is compact in (\mathbb{R}, d_2) , but $[0, 1)$ is not. (Both sets are bounded, but only the first is closed, so the Bolzano-Weierstrass theorem establishes that only the first is compact.) So $f([0, 1])$ is compact but $[0, 1)$ is not compact. We conclude that $f([0, 1]) \neq [0, 1)$.

- (iii) Consider the metric spaces (X, d_X) , (Y, d_Y) , (Z, d_Z) , and $(X \times Y, d_\infty)$ where

$$d_\infty(x, y; x', y') = \max \{d_X(x, x'), d_Y(y, y')\}.$$

Suppose that $f : X \times Y \rightarrow Z$ is continuous. Prove that if $(x_n, y_n) \rightarrow (x^*, y^*)$ inside $(X \times Y, d_\infty)$ then

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} f(x_m, y_n) = f(x^*, y^*).$$

Answer. Since $(x_n, y_n) \rightarrow (x^*, y^*)$ inside $(X \times Y, d_\infty)$, it follows that $x_n \rightarrow x^*$ inside (X, d_X) and $y_n \rightarrow y^*$ inside (Y, d_Y) . Therefore, $\lim_{n \rightarrow \infty} (x_m, y_n) = (x_m, y^*)$ inside $(X \times Y, d_\infty)$.

Since f is continuous and $\lim_{n \rightarrow \infty} (x_m, y_n) = (x_m, y^*)$, we know that $\lim_{n \rightarrow \infty} f(x_m, y_n) = f(x_m, y^*)$ for all m . Similarly, $f(x_m, y^*) \rightarrow f(x^*, y^*)$ since $(x_m, y^*) \rightarrow (x^*, y^*)$. Putting these two conclusions together gives

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} f(x_m, y_n) = \lim_{m \rightarrow \infty} f(x_m, y^*) = f(x^*, y^*).$$

- (iv) Let (X, d) , and (X, d') be two metric spaces. Suppose that both metric spaces have the same open sets, i.e. U is open inside (X, d) if and only if U is open inside (X, d') . Consider any sequence $x_n \in X$. Prove that x_n is convergent inside (X, d) if and only if it is convergent inside (X, d') .

Answer. It suffices to prove one direction only. Swapping the roles of d and d' gives the reverse direction.

Consider the identity function $f(x) = x$ from (X, d) to (X, d') . We will show f is continuous using the open-set characterisation of continuity. Pick any set U that is open (X, d') . By assumption U is open in (X, d) . So $f^{-1}(U) = U$ is open. It follows that f is continuous.

Pick any sequence $x_n \rightarrow_d x^*$. Since f is continuous, $f(x_n) \rightarrow_{d'} f(x^*)$, and hence $x_n \rightarrow_{d'} x^*$.

Another possible answer. It suffices to prove one direction only. Swapping the roles of d and d' gives the reverse direction.

Throughout this proof, we will study open balls using both metrics. We will write $N_r(x)$ when consider open balls using d , and $N'_r(x)$ when using d' .

Suppose $x_n \rightarrow_d x^*$. We will prove that $x_n \rightarrow_{d'} x^*$. Pick any $r' > 0$. Consider the open ball $N'_{r'}(x^*)$. Since $N'_{r'}(x^*)$ is open inside (X, d') , it is also open inside (X, d) . Therefore, there exists some $r > 0$ such that $N_r(x^*) \subseteq N'_{r'}(x^*)$.

Since $x_n \rightarrow_d x^*$, we know that there exists some N such that:

- $d(x_n, x^*) < r$ for all $n > N$, and hence
- $x_n \in N_r(x^*)$ for all $n > N$, and hence
- $x_n \in N'_{r'}(x^*)$ for all $n > N$, and hence
- $d'(x_n, x^*) < r'$ for all $n > N$.

We conclude that $x_n \rightarrow_{d'} x^*$.

(v) Let $X = [-1, 1]$. Recall that $(B(X), d_\infty)$ is the metric space defined by

$$B(X) = \{f : X \rightarrow \mathbb{R} \text{ s.t. } f \text{ is bounded}\}$$

and $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$. We say that a function $g \in B(X)$ is a polynomial if there exist numbers $a_0, a_1, \dots, a_n \in \mathbb{R}$ such that $g(x) = \sum_{i=0}^n a_i x^i$ for all $x \in X$. Let $P(X)$ be the set of all such polynomials. Consider the function $f^* \in B(X)$ defined by

$$f^*(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Prove that there is no sequence $f_n \in P(X)$ such that $f_n \rightarrow f^*$ inside $(B(X), d_\infty)$. *Hint: polynomials are continuous, and the continuous bounded functions form a complete metric space.*

Answer. The hint says that the metric space $(CB(X), d_\infty)$ is complete (we proved this in class) and $P(X) \subset CB(X)$.

Suppose for the sake of contradiction that there exists a sequence $f_n \in P(X)$ such that $f_n \rightarrow f^*$. Since $P(X) \subset CB(X)$, we know that $f_n \in CB(X)$. Since f^* is discontinuous, we know that $f^* \notin CB(X)$. Since $(CB(X), d_\infty)$ is complete, it is a closed subset of $(B(X), d_\infty)$. Since $CB(X)$ is closed, $f^* \in CB(X)$, which is a contradiction.

(vi) Consider the compact metric spaces (X, d_X) , (Y, d_Y) , and $(X \times Y, d_\infty)$ where

$$d_\infty(x, y; x', y') = \max \{d_X(x, x'), d_Y(y, y')\}.$$

Consider a function $f : X \times Y \rightarrow X \times Y$. Suppose that for all $x \in X$, the function $g(y) = f_2(x, y)$ is a contraction on (Y, d_Y) , and similarly $f_1(\cdot, y)$ is a contraction on (X, d_X) for all $y \in Y$. Prove that f has a fixed point in $(X \times Y, d_\infty)$.

Answer. Pick any $(x_0, y_0) \in X \times Y$, and let $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$. Since X is compact, x_n has a convergent subsequence $a_n \in X$ that converges to some point $x^* \in X$. Similarly, y_n has a convergent subsequence $b_n \in Y$ converging to $y^* \in Y$.

Now, consider the sequence (x_n, y^*) . Since $f_1(\cdot, y^*)$ is a contraction, x_n converges to some fixed point $\hat{x} \in X$ by Banach's fixed point theorem, i.e. $f_1(\hat{x}, y^*) = \hat{x}$. (Recall that compact spaces are complete.) Similarly, y_n converges to some point $\hat{y} \in Y$.

Since a_n is a subsequence of x_n , and $a_n \rightarrow x^*$ and $x_n \rightarrow \hat{x}$, we deduce that $x^* = \hat{x}$. By similar logic, $y^* = \hat{y}$.

So $x^* = f_1(x^*, y^*)$ and $y^* = f_2(x^*, y^*)$, and we conclude that (x^*, y^*) is a fixed point of f .

- (vii) Suppose a country is either in a boom ($x = 1$) or recession ($x = 0$), which affects its tax revenue of t_x . Recessions occur each period with probability p . It has savings of a . It can use a and t_x to finance government spending g and future savings a' at interest rate $1/r - 1$. Government programmes g give a utility $u(g)$ each period which is discounted by β , where u is increasing, continuous, concave and bounded. Its Bellman equation is

$$V(x, a) = \sup_{g, a' \geq 0} u(g) + \beta[pV(0, a') + (1-p)V(1, a')] \\ \text{s.t. } g + ra' = t_x + a.$$

- (a) Formulate an appropriate domain of the corresponding Bellman operator.
Hint: ensure it is a complete metric space.

Answer. The state space is $S = \{0, 1\} \times \mathbb{R}_+$. The Bellman operator operates on bounded value functions from the set $B(S)$. With the sup metric, $(B(S), d_\infty)$ is a complete metric space. (We proved this in class.)

- (b) Prove that the Bellman operator is a contraction.

Answer. The Bellman operator $F : B(S) \rightarrow B(S)$ is

$$F(V)(x, a) = \sup_{a' \in [0, (t_x + a)/r]} u(t_x + a - ra') + \beta[pV(0, a') + (1-p)V(1, a')]$$

Pick any $V, W \in B(S)$. Then,

$$\begin{aligned}
& F(V)(x, a) \\
&= \sup_{a' \in [0, (t_x + a)/r]} u(t_x + a - ra') + \beta[pV(0, a') + (1 - p)V(1, a')] \\
&= \sup_{a' \in [0, (t_x + a)/r]} u(t_x + a - ra') + \beta[pV(0, a') + (1 - p)V(1, a')] \\
&\quad + \beta[pW(0, a') + (1 - p)W(1, a') - pW(0, a') + (1 - p)W(1, a')] \\
&\leq \left[\sup_{a' \in [0, (t_x + a)/r]} u(t_x + a - ra') + \beta[pW(0, a') + (1 - p)W(1, a')] \right] \\
&\quad + \left[\sup_{a' \in \mathbb{R}_+} \beta[pV(0, a') + (1 - p)V(1, a') - pW(0, a') + (1 - p)W(1, a')] \right] \\
&= F(W)(x, a) + \left[\sup_{a' \in \mathbb{R}_+} \beta[pV(0, a') + (1 - p)V(1, a') - pW(0, a') + (1 - p)W(1, a')] \right] \\
&\leq F(W)(x, a) + \beta \left[p \sup_{a' \in \mathbb{R}_+} [V(0, a') - W(0, a')] + (1 - p) \sup_{a' \in \mathbb{R}_+} [V(1, a') - W(1, a')] \right] \\
&\leq F(W)(x, a) + \beta [pd_\infty(V, W) + (1 - p)d_\infty(V, W)] \\
&= F(W)(x, a) + \beta d_\infty(V, W).
\end{aligned}$$

Rearranging, we get

$$F(V)(x, a) - F(W)(x, a) \leq \beta d_\infty(V, W).$$

Repeating the logic above with V and W swapped, we get

$$F(W)(x, a) - F(V)(x, a) \leq \beta d_\infty(V, W).$$

Combining (taking the maximum of the two), we get:

$$|F(V)(x, a) - F(W)(x, a)| \leq \beta d_\infty(V, W).$$

Since the above inequality holds for all (x, a) , it also holds when taking the supremum:

$$\sup_{(x, a) \in S} |F(V)(x, a) - F(W)(x, a)| \leq \beta d_\infty(V, W).$$

We conclude that $d_\infty(F(V), F(W)) \leq \beta d_\infty(V, W)$ and F is a contraction of degree β .

- (viii) Suppose a hospital uses n nurses and v ventilators to treat Coronavirus infections, which trade at prices w and p respectively. Both nurses and respirators are needed to save lives. The hospital saves $f(n, v)$ lives, whom it values at x each. The hospital solves the following problem:

$$\sup_{n, v} xf(n, v) - wn - pv.$$

Assume that f is continuous and has constant returns to scale, i.e. $f(an, av) = af(n, v)$ for all (n, v) and all $a > 0$. Prove that there is an optimal v/n ratio.

Answer. We reformulate the problem by selecting $(n, v) = (az, a(1 - z))$:

$$\begin{aligned}
& \sup_{n,v} xf(n, v) - wn - pv \\
&= \sup_{a \in \mathbb{R}_+, z \in [0,1]} xf(az, a(1 - z)) - waz - pa(1 - z) \\
&= \sup_{a \in \mathbb{R}_+, z \in [0,1]} xaf(z, 1 - z) - waz - pa(1 - z) \\
&= \sup_{a \in \mathbb{R}_+} \sup_{z \in [0,1]} xaf(z, 1 - z) - waz - pa(1 - z) \\
&= \sup_{a \in \mathbb{R}_+} a \sup_{z \in [0,1]} xf(z, 1 - z) - wz - p(1 - z).
\end{aligned}$$

Now the inner problem,

$$\sup_{z \in [0,1]} xf(z, 1 - z) - wz - p(1 - z)$$

has a compact menu $[0, 1]$ and a continuous objective, so there is an optimal choice z^* . By assumption, $z^* \neq 0$ and $z^* \neq 1$ are optimal – both nurses and ventilators are needed to save lives. So there is an optimal v/n ratio, namely $\frac{1-z^*}{z^*}$.

38: Micro 1, May 2020

Consider an economy consisting of two types of firms – startups and farms, and two types of household – engineer and unskilled. A startup produces a completely new type of vegan snack food. It has two possible ways to operate. First, it could use engineers and vegetables only. Alternatively, it could use 5000 engineer-hours to develop a more efficient constant-returns-to-scale technology that transforms engineering labour, unskilled labour, and vegetables into vegan snack food. In the second case, engineers have a higher marginal product than unskilled workers. A farm hires workers to make vegetables (either type is equally productive). Some households are endowed with engineering labour, and the other households are endowed with unskilled labour. Households sell their labour inelastically, and consume both vegetables and vegan snack food.

- (i) Formulate a competitive model of the vegan snack food, vegetables, engineering labour, and unskilled labour markets. Formulate the startup's problem using a Bellman equation involving a choice between the two alternatives.

Comments. Common mistakes included

- Assuming that all households are identical (engineers are different).
- Having a different number of wages than labour market clearing conditions.
- Not modelling the start-up technology choice correctly. Many students either assumed that both technologies are adopted, or studied two different models, one for each technology.

Answer. Let n_e be the number of engineers, n_m be the number of unskilled workers, $n = n_e + n_m$ be the number of households, π_f the farm profits, and π_s the start-up profits.

Engineer Households. Each engineer household is endowed with h_e hours of labour, which it sells at wage w_e . It also receives an equal share $(\pi_f + \pi_s)/n$ of the firms' profits. It chooses how much vegan snacks s_e and vegetables t_e buy, at prices p and q respectively. This gives the household a utility of $u_e(s_e, t_e)$. The household's utility maximisation problem is

$$\begin{aligned} \max_{s_e, t_e} & u_e(s_e, t_e) \\ \text{s.t.} & ps_e + qt_e = w_e h_e + (\pi_f + \pi_s)/n. \end{aligned}$$

Unskilled Households. Each unskilled household is endowed with h_m hours of labour, which it sells at wage w_m . It also receives an equal share $(\pi_f + \pi_s)/n$ of the firms' profits. It chooses how much vegan snacks s_m and vegetables t_m buy, at prices p and q respectively. This gives the household a utility of $u_m(s_m, t_m)$. The household's utility maximisation problem is

$$\begin{aligned} \max_{s_m, t_m} & u_m(s_m, t_m) \\ \text{s.t.} & ps_m + qt_m = w_m h_m + (\pi_f + \pi_s)/n. \end{aligned}$$

Farm. The farm chooses how many engineers H_{fe} and unskilled workers H_{fm} to hire, which leads to $f(H_{fe} + H_{fm})$ units of vegetables. The farm's profit maximisation problem is

$$\pi_f(q; w_e, w_m) = \max_{H_{fe}, H_{fm}} qf(H_{fe} + H_{fm}) - w_e H_{fe} - w_m H_{fm}.$$

Startup. If the start-up uses the first alternative ($a = 1$), it hires H_{se} engineers and purchases T_s vegetables and produces $g_1(H_{se}, T_s)$ vegan snacks. In this case, its profits would be

$$\pi_{s1}(p; w_e, q) = \max_{H_{se}, T_s} pg_1(H_{se}, T_s) - w_e H_{se} - qT_s.$$

If the start-up uses the second alternative ($a = 2$), and already has its plans ready, then it hires H_{se} engineers, H_{sm} unskilled workers, and purchases T_s vegetables and produces $g_2(H_{se}, H_{sm}, T_s)$ vegan snacks. In this case, its profits would be

$$\pi_{s2}(p; w_e, w_s, q) = \max_{H_{se}, H_{sm}, T_s} pg_2(H_{se}, H_{sm}, T_s) - w_e H_{se} - w_m H_{sm} - qT_s.$$

The start-up's profit function is

$$\pi(p; w_e, w_s, q) = \max \{ \pi_{s1}(p; w_e, q), \pi_{s2}(p; w_e, w_s, q) - 5000w_e \}.$$

Equilibrium. The prices (p, q, w_e, w_m) and quantities

$$(s_e, t_e, s_m, t_m, H_{fe}, H_{fm}, H_{se}, H_{sm}, T_s),$$

constitute an equilibrium if the quantities are optimal choices in the problems above, and all markets clear:

$$\begin{aligned} n_e s_e + n_m s_m &= I(a = 1)g_1(H_{se}, T_s) + I(a = 2)g_2(H_{se}, H_{sm}, T_s) \\ n_e t_e + n_m t_m + T_s &= f(H_{fe} + H_{fm}) \\ n_e h_e &= H_{fe} + H_{se} + I(a = 2)5000 \\ n_m h_m &= H_{fm} + I(a = 2)H_{sm}. \end{aligned}$$

- (ii) Is it possible for there to be excess supply of both types of food and both types of labour (at non-equilibrium prices)?

Answer. No. By Walras' law, if there is excess supply in one market, there must be excess demand in another market.

- (iii) Prove that in every equilibrium, unskilled workers receive a smaller or equal wage than the engineers.

Comment. This question was deceptively difficult – it requires carefully considering several possibilities (whether any engineers take on unskilled tasks, and whether the start-up invests in the better technology).

Many students got confused, because first-order conditions only apply to “interior” choices, i.e. the marginal benefit of hiring an engineer only equals the marginal cost when a firm actually hires some engineers. (If the marginal cost is higher at zero, then the firm hires none.)

Answer. If unskilled workers received a higher wage, then

- the farm would only hire engineers, and
- the startup would do either of the following:
 - Pursue alternative $a = 1$, and only hire engineers. In this case, no unskilled workers are hired, violating the market clearing condition for unskilled workers. This can not be an equilibrium.
 - Pursue alternative $a = 2$, and hire both. The firm's first-order conditions with respect to H_{se} and H_{sm} are:

$$p \frac{\partial}{\partial H_{se}} g_2(H_{se}, H_{sm}, T_s) = w_e$$

$$p \frac{\partial}{\partial H_{sm}} g_2(H_{se}, H_{sm}, T_s) = w_m.$$

Dividing the first by the second, we get

$$\frac{\frac{\partial}{\partial H_{se}} g_2(H_{se}, H_{sm}, T_s)}{\frac{\partial}{\partial H_{sm}} g_2(H_{se}, H_{sm}, T_s)} = \frac{w_e}{w_m}.$$

Since engineers have higher marginal productivity, the left side is bigger than one. Therefore, the right side is bigger than one, and engineers would have to have higher wages than unskilled workers.

- (iv) Suppose the startup develops the more efficient technology. Prove that the startup responds to an engineering wage increase by hiring fewer engineers.

Answer. Recall that the start-ups profit function in this case is:

$$\pi_{s2}(p; w_e, w_s, q) = \max_{H_{se}, H_{sm}, T_s} pg_2(H_{se}, H_{sm}, T_s) - w_e H_{se} - w_m H_{sm} - qT_s.$$

By the envelope theorem,

$$\begin{aligned} & \frac{\partial}{\partial w_e} \pi_{s2}(p; w_e, w_s, q) \\ &= \left[\frac{\partial}{\partial w_e} \{pg_2(H_{se}, H_{sm}, T_s) - w_e H_{se} - w_m H_{sm} - qT_s\} \right]_{\text{at optimal } (H_{se}, H_{sm}, T_s)} \\ &= [-H_{se}]_{\text{at optimal } (H_{se}, H_{sm}, T_s)} \\ &= -H_{se}(p; w_e, w_s, q). \end{aligned}$$

Now, π_{s2} is a convex function, because it is the upper envelope of linear (and hence convex) functions of prices – one function per choice of (H_{se}, H_{sm}, T_s) . Therefore, the left side of the equation above is (weakly) increasing.

It follows that the right side is decreasing, and hence the firm hires fewer engineers.

- (v) Suppose that there is only one equilibrium, and it involves the startup pursuing the first alternative. The government would like to increase entrepreneurial activity, i.e. to ensure that the start-up pursues the second alternative. When is it possible to design a lump-sum tax scheme to do this?

Comment. There were two common mistakes:

- Some students thought that lump-sum transfers would not affect the equilibrium allocation, because the question said there is only one equilibrium. The question meant there is only one equilibrium for the given model parameters. Lump-sum transfers are a change in model parameters.
- Some students argued that any equilibrium in which the start-up invests in the new technology would make everyone better off, and would therefore be impossible by the first welfare theorem. This logic is unsound – if there is an efficient allocation involving the second alternative, then it involves making some people worse off.

Answer. If there is an efficient allocation involving the second alternative, then it is possible, by the second welfare theorem. Since developing new technology is efficient if and only if large amounts of vegan snacks are to be produced, this boils down to: is there an efficient allocation involving a large amount of vegan snacks?

There are many different model parameters that could lead to this being efficient. For example, suppose households consuming small quantities of everything consider snacks and vegetables to be perfect substitutes, but when they consume large quantities, they have a higher marginal utility of snacks. In this case, efficient unequal allocations might involve more snack consumption than egalitarian allocations.

- (vi) * Formulate the excess demand function of the economy, and use it to express the market-clearing conditions.

Answer. Let $P = (p, q, w_e, w_s)$ be the vector of prices. The excess demand function is

$$z(P) = \begin{bmatrix} n_e s_e(P) + n_m s_m(P) \\ n_e t_e(P) + n_m t_m(P) \\ n_e h_e(P) \\ n_m h_m(P) \end{bmatrix} - \begin{bmatrix} \begin{cases} g_1(H_{se}(P), T_s(P)) & \text{if } a(P) = 1 \\ g_2(H_{se}(P), H_{sm}(P), T_s(P)) & \text{if } a(P) = 2 \end{cases} \\ f(H_{fe}(P) + H_{fm}(P)) \\ H_{fe}(P) + H_{se}(P) + I(a(P) = 2)5000 \\ H_{fm}(P) + I(a(P) = 2)H_{sm}(P) \end{bmatrix}.$$

The market clearing conditions are satisfied if and only if

$$z(P) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

39: AME, December 2020

Part A

Suppose all households are endowed with broken mobile phones. Half of the households have new model broken phones, and half have old model broken phones. They sell their broken phones. Repair shops hire workers and buy broken phones, and sell working phones (both models). Factories hire workers and make (working) new model phones. Households choose how many phones of each type to buy, and how much labour to supply.

- (i) Write down a competitive model of the four phone markets (broken/working, new/old model) and the labour market.

Comment. Common mistakes include:

- Treating all households as identical. In particular, households endowed with old phones will make different choices than those with new phones, so this needs to be reflected in the notation.
- Assuming that the repair firm allocates the same worker-hours to repairing both old and new phones, i.e. one worker does two jobs at the same time.
- Having two labour markets but one wage, or the other way around.

Answer. Households. Household $h \in H$ is endowed with e_h^m broken phones of model $m \in \{0, 1\}$ and 1 unit of time. It sells its broken phones of model m at price q^m , and ℓ_h units of labour at price w . It buys x_h^m phones of model m at price p^m . This gives household h a utility of $u(x_h^0, x_h^1, 1 - \ell_h)$. Its utility maximisation problem is

$$\begin{aligned} & \max_{x_h^0, x_h^1, \ell_h} u(x_h^0, x_h^1, 1 - \ell_h) \\ \text{s.t. } & p^0 x_h^0 + p^1 x_h^1 = w \ell_h + q^0 e_h^0 + q^1 e_h^1 + \frac{1}{|H|} \left(\sum_{s \in S} \pi_s + \pi \right). \end{aligned}$$

Repair shops. Repair shop $s \in S$ buys E_s^0 and E_s^1 broken phones (old and new respectively). It hires L_s^0 workers to repair old phones and L_s^1 workers to repair new phones. It produces $f^0(E_s^0, L_s^0)$ old phones and $f^1(E_s^1, L_s^1)$ new phones. Its profit function is

$$\pi_s(p^0, p^1; q^0, q^1, w) = \max_{E_s^0, E_s^1, L_s^0, L_s^1} p^0 f^0(E_s^0, L_s^0) + p^1 f^1(E_s^1, L_s^1) - q^0 E_s^0 - q^1 E_s^1 - w(L_s^0 + L_s^1).$$

Factory. The factory hires L workers, and produces $g(L)$ new phones. Its profit function is

$$\pi(p^1; w) = \max_L p^1 g(L) - wL.$$

Equilibrium. Prices (q^0, q^1, p^0, p^1, w) and quantities $(x_h^0, x_h^1, \ell_h, E_s^0, E_s^1, L_s^0, L_s^1, L)$ constitute an equilibrium if the quantities solve the respective optimisation prob-

lems above, and all markets clear, i.e.

$$\begin{aligned}\sum_s E_s^0 &= \sum_h e_h^0 \\ \sum_s E_s^1 &= \sum_h e_h^1 \\ \sum_h x_h^0 &= \sum_s f(E_s^0, L_s^0) \\ \sum_h x_h^1 &= g(L) + \sum_s f(E_s^1, L_s^1) \\ \sum_h \ell_h &= L + \sum_s L_s^0 + \sum_s L_s^1.\end{aligned}$$

(ii) Recall $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly concave if for all $t \in (0, 1)$, and all $x, x' \in \mathbb{R}^n$,

$$tf(x) + (1-t)f(x') < f(tx + (1-t)x').$$

Suppose there are 10 repair shops, and their production function is strictly concave. Prove that all 10 repair shops repair the same number of phones.

Answer. All 10 repair shops solve the same problem, namely

$$\max_{E^0, E^1, L^0, L^1} p^0 f^0(E^0, L^0) + p^1 f^1(E^1, L^1) - q^0 E^0 - q^1 E^1 - w(L^0 + L^1).$$

In fact, this problem can be split into two separate problems,

$$\max_{E^m, L^m} p^m f^m(E^m, L^m) - q^m E^m - wL^m,$$

one for $m = 0$ and for $m = 1$. Since f^m is strictly concave, and the remaining terms are linear, the objective is strictly concave. So each problem has a unique solution. Therefore, each repair shop makes the same choice.

(iii) Suppose the phone manufacturer buys all of the repair shops. Write down the conglomerate's profit function using a Bellman equation.

Answer.

$$\Pi(p^0, p^1; q^0, q^1, w) = \pi(p^1, w) + 10\pi_1(p^0, p^1; q^0, q^1, w).$$

(iv) Calculate the marginal repair shop profit of a wage increase.

Answer. By the envelope theorem,

$$\begin{aligned}\frac{\partial}{\partial w} \pi_s(p; q, w) &= \left(\frac{\partial}{\partial w} \left[\sum_m \{p^m f^m(E_s^m, L_s^m) - q^m E_s^m - wL_s^m\} \right] \right)_{E_s=E_s(p; q, w), L_s=L_s(p; q, w)} \\ &= \left(- \sum_m L_s^m \right)_{E_s=E_s(p; q, w), L_s=L_s(p; q, w)} \\ &= -L_s^0(p; q, w) - L_s^1(p; q, w).\end{aligned}$$

Part B

- (i) Let $f_n(x) = \frac{x}{n}$ and $A = \{f_n : n \in \mathbb{N}\}$. Is A a compact set inside $(B[0, 1], d_\infty)$?

Comment. Most students made the wrong guess – A is not compact.

Answer. No. The sequence $a_n(x) = \frac{x}{n}$ converges to $a^*(x) = 0$. But $a^* \notin A$. So A is not closed in $(B[0, 1], d_\infty)$, and hence not compact.

- (ii) Let $F = B((0, 1))$ be the set of bounded functions with domain $(0, 1)$ and co-domain \mathbb{R} . Consider the sequence of functions $f_n \in F$ defined by $f_n(x) = \frac{x}{n}$. Find a metric d such that (F, d) is a metric space and f_n is not convergent.

Answer. Let d be the discrete metric, i.e. $d(x, y) = 1$ if $x \neq y$, and 0 otherwise. Then $d(f_n, f_m) = 1$ for all $n \neq m$. So f_n is not a Cauchy sequence, and hence it is not convergent.

- (iii) Find a counter-example to this false conjecture. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Consider the metric space (Z, d_Z) , where $Z = X \times Y$ and

$$d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y').$$

If A is a closed set inside (Z, d_Z) , then $A_X = \{x : (x, y) \in A\}$ is a closed set in (X, d_X) .

Comment. Lots of students got stuck on this one. The counter-examples weren't really counterexamples. For example, many students proposed a set A that wasn't a closed set.

Answer. Consider $(X, d_X) = (Y, d_Y) = (\mathbb{R}, d_2)$ and $A = \{(x, y) \in X \times Y : xy \geq 1\}$. Let $f(x, y) = xy$. Notice that $A = f^{-1}([1, \infty))$. Now A is the pre-image of a continuous function on a closed set. So A is closed. But $A_X = \mathbb{R} \setminus \{0\}$, which is not closed in (\mathbb{R}, d_2) .

- (iv) Consider the metric spaces (X, d_X) , (Y, d_Y) and (Z, d_Z) and sets A and A_X defined in the previous question. Prove that if A is a compact set inside (Z, d_Z) , then A_X is a compact set inside (X, d_X) .

Answer. Consider the function $f : X \times Y \rightarrow X$ defined by $f(x, y) = x$.

First, notice that f is continuous. If $(x_n, y_n) \rightarrow (x^*, y^*)$, then $d_2(x_n, x^*) + d_2(y_n, y^*) \rightarrow 0$, so $d_2(x_n, x^*) \rightarrow 0$ and hence $f(x_n, y_n) = x_n \rightarrow x^* = f(x^*, y^*)$.

Second, notice that $A_X = f(A)$. Since A is compact and f is continuous, we conclude that A_X is compact.

- (v) Let (X, d) be a compact metric space, A be a closed set, B be an open set with $A \subseteq B$. Prove that there exists a continuous function $f : X \rightarrow \mathbb{R}$ such that (i) $f(x) = 0$ for all $x \in A$, and (ii) $f(x) > 1$ for all $x \notin B$. *Hint: You may make use of the following theorem without proving it: d is continuous.*

Comment. This question had a major mistake in it: B is supposed to be open, not closed. This theorem is related to Urysohn's lemma, which is included in many text books.

Answer. Let $g(x) = \min_{a \in A} d(x, a)$ and let $r = \min_{a, x \in A \times (X \setminus B)} d(a, x)$. Finally, let $f(x) = 2 \frac{g(x)}{r}$. We will prove that f exists and satisfies the both properties.

First, we must prove that g exists. Pick any $x \in X$. Now $a \mapsto d(x, a)$ is continuous (this was a homework question). Since there is a continuous objective with a compact domain A , the extreme value theorem implies that $g(x)$ exists. We conclude that g exists.

Second, we prove that r exists and $r > 0$. Notice that A and $X \setminus B$ are closed sets inside (X, d) . It follows that $A \times (X \setminus B)$ is a closed set inside (X^2, d_∞) where $d_\infty(x, y; x', y') = \max \{d(x, x'), d(y, y')\}$. (Proof omitted.) Similarly, (X^2, d_∞) is a compact metric space. Since $A \times (X \setminus B)$ is a closed set inside a compact space, it follows that $A \times (X \setminus B)$ is compact. By the extreme value theorem, there exists some (a, x) that minimises d on this compact set. Thus, $r = d(a, x)$ exists. We can rule out $r = 0$, because this would imply $a = x$ which contradicts $a \in A$ and $x \in X \setminus B$. We conclude that r exists and $r > 0$.

Since g and $r > 0$ exist, f also exists. The first property is satisfied, because $\min_{a \in A} d(x, a) = d(x, x) = 0$ for all $x \in A$. The second property is satisfied, because $g(x) \geq r$, so $f(x) \geq 2 > 1$.

- (vi) Consider the set of wealth distributions (Lorenz curves),

$$X = \{f \in C(\mathbb{R}, [0, 1]) : f \text{ is a weakly increasing}\}.$$

Prove that (X, d_∞) is a complete metric space.

Comment. Some students forgot to mention completeness.

Answer. We proved in class that $(C(\mathbb{R}, [0, 1]), d_\infty)$ is a complete metric space. (Note that the restricted codomain $[0, 1]$ eliminates unbounded functions.) Thus it suffices to show that X is a closed set. Suppose $f_n \in X$ converges to f^* . We need to show that $f^* \in X$, i.e. that f^* is weakly increasing.

Pick any $a, b \in \mathbb{R}$ with $a < b$. To prove that f^* is weakly increasing, we will show that $f^*(a) \leq f^*(b)$. Since each f_n is weakly increasing, we know that $f_n(b) - f_n(a) \geq 0$ for all n . Since $f_n \rightarrow f^*$, we know that

$$f_n(b) - f_n(a) \rightarrow f^*(b) - f^*(a).$$

Since $f_n(b) - f_n(a)$ is a sequence inside \mathbb{R}_+ which is a closed set, we deduce that its limit $f^*(b) - f^*(a) \geq 0$. So $f^*(b) > f^*(a)$, as required.

- (vii) Consider the set of wealth distributions X from the previous question. Suppose that today's wealth distribution is $f_0 \in X$. Margaret Thatcher's poll tax transforms year n 's distribution, f_n , into $f_{n+1} = T(f_n)$ the following year, where T is a contraction. Robin Hood does not like Margaret Thatcher's T function, so he tries to undo it by replacing f_0 with \hat{f}_0 . Margaret Thatcher's T function applies thereafter, i.e. $\hat{f}_{n+1} = T(\hat{f}_n)$. Explain why Robin Hood's intervention is ineffective in the long run.

Answer. Since $T : X \rightarrow X$ is a contraction on the complete metric space (X, d_∞) , Banach's fixed point theorem applies. The theorem implies that T has a unique

fixed point, f^* . It also implies that both $f_n \rightarrow f^*$ and $\hat{f}_n \rightarrow f^*$. So in the long run, the wealth distributions with and without Robin Hood converge to the same thing.

- (viii) Let $a \in [0, 1]$ be the quality of a factory, $r \in [1.01, 2]$ be the interest rate, w be wages, $h \in [0, 1]$ be hours of work. Every period, the factory has to decide whether to shutdown (permanently), and how much work to put into maintenance of the factory. Its Bellman equations are

$$V(a, r) = \frac{1}{r} \max \{0, W(a, r)\}$$

$$W(a, r) = \max_{h \in [0, 1]} a - wh + V(\sqrt{ah}, r).$$

Comment. This is a simplified version of Kiyotaki, Moore and Zhang's (2020) model.

- (a) Reformulate the factory's problem using a single Bellman equation with V on both sides.

Answer. Substituting the second equation into the first gives

$$V(a, r) = \frac{1}{r} \max \left\{ 0, \max_{h \in [0, 1]} a - wh + V(\sqrt{ah}; r) \right\}.$$

- (b) What is the corresponding Bellman operator? Don't forget to specify the metric space for the domain and co-domain.

Answer. The Bellman operator $T : B([0, 1] \times [1.01, 2]) \rightarrow B([0, 1] \times [1.01, 2])$ is defined by

$$T(V)(a, r) = \frac{1}{r} \max \left\{ 0, \sup_h a - wh + V(\sqrt{ah}; r) \right\}.$$

Distances in the domain and codomain can be measured by

$$d_\infty(f, g) = \sup_{a, r} |f(a, r) - g(a, r)|.$$

- (c) Assume that the domain is a complete metric space, and the operator is a contraction. Prove that V is strictly decreasing in r .

Answer. First, let $X = \{f \in B([0, 1] \times [1.01, 2], \mathbb{R}_+) : f(a, r) \text{ is weakly decreasing in } r\}$. Now, X is a closed subset, so (X, d_∞) is a complete metric space.

Second, if $V \in X$ then $T(V) \in X$. To see this, notice that $\frac{1}{r}$ is strictly decreasing in r , and V is weakly decreasing in r (by assumption), and both are positive. So the product is strictly decreasing and positive.

So T is a contraction on the complete metric space (X, d_∞) . Banach's fixed point theorem implies that T has a unique fixed point $V^* \in X$. In other words, the Bellman equation has a solution V^* which is weakly decreasing in r .

Now, the Bellman equation has a unique solution (since Banach's fixed point theorem applies on the larger space $(B([0, 1] \times [1.01, 2], \mathbb{R}_+), d_\infty)$). So the only solution to the Bellman equation is weakly decreasing in r .

Next, notice that if $V \in X$, then $T(V)$ is strictly greater than 0. Since $V^* = T(V^*)$, it follows that $V^*(a, r) > 0$ for all (a, r) .

Finally, if $V \in X$ is weakly decreasing in r and strictly greater than 0, then $T(V)$ is strictly decreasing in r . Since $V^* = T(V^*)$, we conclude that V^* is strictly decreasing in r , as required.

40: Micro 1, December 2020

Suppose that everyone knows how to make pizza. A household of Russian Jews moves to Edinburgh, and opens a restaurant selling blintzes (a sweet crepe), and a school teaching the locals how to cook blintzes. The course is indivisible, takes a year to complete, and runs in the first year only. The households allocate their time in two years between leisure, studying, and supplying unskilled or skilled (in blintzes) labour. Each household chooses how much of each type of food to eat each year. The Russian household owns the Russian restaurant/school, and the other households hold equal shares in the pizza restaurant. Assume that all households do not value leisure, and have undiscounted utility functions, i.e. they are perfectly patient.

- (i) Formulate a competitive model of the labour, food and education markets over two years.

Comment. Common mistakes were:

- incorrectly distinguishing between the local and Russian households. For instance, the Russian household is already skilled, so it ought not need to study in order to do skilled work in the second period.
- modelling study as a continuous variable, even though the question specified that workers are either skilled or not.
- allowing unskilled workers to do skilled work.
- assuming all households are the same, or all households make the same choices, or even that all local households make the same choices.
- treating blintzes across both periods as a single market with a single price.

Answer. The set of households $H = H_L \cup \{r\}$ consists of the locals H_L and the Russian household r .

Households. Each local household $h \in H_L$ chooses how many year-hours of labour $\ell_{hst} \in [0, 1]$ to supply of skill level $s \in \{0, 1\}$ in time $t \in \{1, 2\}$, whether to study e_h – which takes $E \in [0, 1]$ years of study time, how many blintzes b_{ht} and pizzas c_{ht} to consume. This gives the local household h a utility of

$$\sum_{t=1}^2 u(b_{ht}, c_{ht})$$

Wages of skill s at time t are w_{st} , the price of tuition is x , of blintzes is p_t and of pizzas is q_t pizza (both in time t). Households can only supply skilled labour if they are skilled. Specifically, the Russian household is endowed with a skill level of $k_{r1} = 1$, whereas the other households $h \in H_L$ is endowed with $k_{h1} = 0$. In the second period, the skill level is

$$k_{h2} = \max \{I(h = r), I(e_h = 1)\}.$$

Household h receives dividends of π^h , which is a share of the pizza profits $\pi_c/|H_L|$ for local households, and all of the blintzes profits π_b for the Russian household.

The utility maximisation problem for household h is

$$\begin{aligned} & \max_{\ell_{hst}, e_h, b_{ht}, c_{ht}} \sum_{t=1}^2 u(b_{ht}, c_{ht}) \\ \text{s.t. } & \sum_{t=1}^2 (p_t b_{ht} + q_t c_{ht}) + e_h x = \sum_{(s,t) \in \{0,1\} \times \{1,2\}} w_{st} \ell_{hst} + \pi^h \\ & \text{and } \sum_s \ell_{hst} = 1 \text{ for } t \in \{1, 2\} \\ & \text{and } \ell_{h1t} = 0 \text{ if } k_{ht} = 0 \text{ for } t \in \{1, 2\}. \end{aligned}$$

Blintzes firm/school. The blintzes firm hires L_{bt} skilled workers in each time period to make $f(L_{bt})$ blintzes in time t , and L_e skilled workers to teach $g(L_e)$ students in time 1. Its profit function is

$$\pi_b(p, x; w_{11}, w_{12}) = \max_{L_{b1}, L_e} p_1 f(L_{b1}) + p_2 f(L_{b2}) + x g(L_e) - w_{11}(L_{b1} + L_e) - w_{12} L_{b2}.$$

Pizza firm. The pizza firm hires L_{ct} workers in each time period to make $h(L_{ct})$ pizzas. Its profit function is

$$\pi_c(q; w_{01}, w_{02}) = \max_{L_{c1}, L_{c2}} q_1 h(L_{c1}) + q_2 h(L_{c2}) - w_{01} L_{c1} - w_{02} L_{c2}.$$

Equilibrium. The prices (p_t, q_t, w_{st}, x) and quantities $(\ell_{hst}, e_h, b_{ht}, c_{ht}, L_{bt}, L_e, L_{ct})$ form an equilibrium if the quantities are optimal choices in the problems above, and all markets clear:

$$\begin{aligned} \sum_{h \in H} \ell_{h01} &= L_{c1} \\ \sum_{h \in H} \ell_{h02} &= L_{c2} \\ \sum_{h \in H} \ell_{h11} &= L_{b1} + L_e \\ \sum_{h \in H} \ell_{h12} &= L_{b2} \\ \sum_{h \in H} b_{h1} &= f(L_{b1}) \\ \sum_{h \in H} b_{h2} &= f(L_{b2}) \\ \sum_{h \in H} c_{h1} &= h(L_{c1}) \\ \sum_{h \in H} c_{h2} &= h(L_{c2}) \\ \sum_{h \in H} e_h &= g(L_e) \end{aligned}$$

- (ii) Prove that if the first-period skilled wages increase, then the Blintzes firm trains fewer people in the first period.

Answer. The Russian firm's profit function can be split into its two divisions:

$$\begin{aligned}\pi_{bb}(p; w_{11}, w_{12}) &= \max_{L_{bt}} p_1 f(L_{b1}) + p_2 f(L_{b2}) - w_{11} L_{b1} - w_{12} L_{b2} \\ \pi_{be}(x; w_{11}) &= \max_{L_e} x g(L_e) - w_{11} L_e \\ \pi_b(p, x; w_{11}, w_{12}) &= \pi_{bb}(p; w_{11}, w_{12}) + \pi_{be}(x; w_{11}, w_{12}).\end{aligned}$$

Note that $g : \mathbb{R}_+ \rightarrow \mathbb{N}$ is a step function, so it is not concave. But that does not matter for our purposes – π_{be} is still the upper envelope of a collection of linear functions of prices (one for each choice of L_e). So π_{be} is a convex function.

By the envelope theorem,

$$\begin{aligned}\frac{\partial}{\partial w_{11}} \pi_{be}(x; w_{11}, w_{12}) &= \left[\frac{\partial}{\partial w_{11}} (xg(L_e) - w_{11}L_e) \right]_{L_e=L_e(x; w_{11}, w_{12})} \\ &= [-L_e]_{L_e=L_e(x; w_{11}, w_{12})} \\ &= -L_e(x; w_{11}, w_{12}).\end{aligned}$$

Since π_{be} is a convex function, the left side is weakly increasing in w_{11} . We deduce that the right side is also weakly increasing, and that teaching hours L_e is weakly decreasing in w_{11} . We conclude that the number of students $g(L_e)$ is weakly decreasing in w_{11} .

- (iii) Suppose that in equilibrium, at least one household studies blintzes. Prove that more blintzes are produced in the second period than the first.

Comment. It was common to skip important steps, such as how we know that all skilled workers might blintzes in the second period.

Answer. In the first period, the Russian household splits its labour between skilled and unskilled, and not all of that skilled labour is allocated to cooking blintzes (assuming some labour is needed to cook and teach). So $L_{b1} < 1$. Studying blintzes in the first period is worthwhile only if skilled labour is paid more in the second period than unskilled labour, i.e. $w_{12} > w_{02}$. So in the second period, the Russian household specialises in skilled labour ($\ell_{r12} = 1$), all of which is allocated to making blintzes. So $L_{b2} \geq 1$. We conclude that $L_{b1} < L_{b2}$, and hence the supply of blintzes is higher in the second period, i.e. $f(L_{b1}) < f(L_{b2})$ assuming f is strictly increasing.

- (iv) Reformulate the local households' problem into a dynamic programming problem in which labour and education are chosen first, and consumption choices are buried inside a value function.

Comment. A common mistake was to bury second-period choices inside a value function (rather than both periods' consumption choices).

Answer. Let V be the value of income, i.e.

$$V(m; p, q) = \max_{b_t, c_t} u(b_1, c_1) + u(b_2, c_2)$$

$$\text{s.t. } p_1 b_1 + p_2 b_2 + q_1 c_1 + q_2 c_2 = m.$$

The household's net income is

$$m(e, p, q, x) = e[(1 - E)w_{01} + w_{12} - x] + (1 - e)[w_{01} + w_{02}].$$

Then the household's problem can be reformulated as

$$\max_{e \in \{0,1\}} V(m(e, p, q, x); p, q).$$

- (v) Prove that in every equilibrium in which both pizza and blintzes are consumed, the local households eat the same food as each other (regardless of the amount of study they do). Hint: assume the utility function is strictly concave.

Comment. A common mistake was to assume that every local household makes the same choices. Many students did not explain why all locals have the same income.

Answer. All local households have the same amount of leisure (0), and all are indifferent between studying or not, i.e. $m(1, p, q, x) = m(0, p, q, x)$. Therefore, all local households face the same (m, p, q) state variables when choosing (b, c) . This problem has a unique solution if u is strictly concave. So all local households choose the same food quantities, (b, c) .

- (vi) The locals notice that the Russian's firm is making a huge profit. They want to restore perfect equality to Edinburgh. They propose a 100% tax on all firms' profits, and distributing these equally among the households. Would this deliver an efficient and equal equilibrium?

Comment. A common mistake was to attempt to apply the second welfare theorem. In fact, the second welfare theorem is unhelpful here, because it operates in the reverse direction. It begins from a desired allocation, and constructs appropriate lump-sum transfers. This question begins with a tax proposal (which are not lump-sum), and investigates the welfare consequences of the proposal.

Answer. This 100% tax is equivalent to reallocating ownership of the firms to the households in an egalitarian manner. By the first welfare theorem, all resulting equilibrium allocations would be efficient. However, the allocations are not egalitarian: the Russian household earns higher wages in the first period, as it is the only skilled household.

41: AME, May 2021

Part A

Suppose that half of the houses in a city are in polluted areas. Residents in polluted areas suffer health problems, and can only do unskilled work. Apart from this problem, all workers can do skilled and unskilled work. A firm hires skilled and unskilled workers to make furniture. Workers are endowed with a house and hours which they can sell. Workers can buy houses and furniture. Workers can not live together (due to fire regulations).

- (i) Formulate a competitive model of the housing, labour and furniture markets.

Answer.

Households. Household $h \in H$ is endowed with a_h clean homes and b_h polluted homes, which it trade at prices p_a and p_b respectively. Household h purchases $A_h \in \{0, 1\}$ clean homes and $B_h \in \{0, 1\}$ polluted homes, so that $A_h + B_h = 1$. The household supplies 1 unit of labour at the wage w_{A_h} , and buys furniture f_h at price p_f . The household receives a share $\frac{\pi_f}{|H|}$ of the furniture firm's profits. The household's utility is $u(A_h, B_h, f_h)$. The household's utility maximisation problem is

$$\begin{aligned} & \max_{A_h \in \{0,1\}, B_h \in \{0,1\}, f_h} u(A_h, B_h, f_h) \\ & \text{s.t. } A_h + B_h = 1 \\ & p_a A_h + p_b B_h + p_f f_h = w_{A_h} + p_a a_h + p_b b_h + \frac{\pi_f}{|H|}. \end{aligned}$$

Furniture firm. The firm hires L_0 unskilled and L_1 skilled workers and produces $g(L_0, L_1)$ items of furniture. Its profit function is

$$\pi_f(p_f, w_0, w_1) = \max_{L_0, L_1} p_f g(L_0, L_1) - w_0 L_0 - w_1 L_1.$$

Equilibrium. Prices $(w_0, w_1, p_a, p_b, p_f)$ and quantities $(A_h, B_h, f_h, L_0, L_1)$ form an equilibrium if these quantities solve the households' and firm's problems above, and all markets clear, i.e.

$$\begin{aligned} \sum_{h \in H} A_h &= L_1 \\ \sum_{h \in H} (1 - A_h) &= L_0 \\ \sum_{h \in H} A_h &= \sum_{h \in H} a_h \\ \sum_{h \in H} B_h &= \sum_{h \in H} b_h \\ \sum_{h \in H} f_h &= g(L_0, L_1). \end{aligned}$$

- (ii) Reformulate the worker's problem using a Bellman equation with a housing choice, and with the other choices buried inside a value function.

Answer.

$$\begin{aligned} & \max_{A_h \in \{0,1\}, B_h \in \{0,1\}, m_h} V(A_h, B_h, m_h) \\ & \text{s.t. } A_h + B_h = 1 \\ & p_a A_h + p_b B_h + m_h = w_{A_h} + p_a a_h + p_b b_h + \frac{\pi_f}{|H|}. \end{aligned}$$

where $V(A_h, B_h, m_h) = u(A_h, B_h, m_h/p_f)$.

- (iii) Write down a formula for the marginal profit of a skilled wage increase.

Answer. By the envelope theorem,

$$\begin{aligned} \frac{\partial \pi_f(p_f, w_0, w_1)}{\partial w_1} &= \left[\frac{\partial}{\partial w_1} \{p_f g(L_0, L_1) - w_0 L_0 - w_1 L_1\} \right]_{L=L(p_f, w_0, w_1)} \\ &= [-L_1]_{L=L(p_f, w_0, w_1)} \\ &= -L_1(p_f, w_0, w_1). \end{aligned}$$

- (iv) Suppose that the firm's production function is strictly concave. Prove that the firm has at most one optimal choice.

Answer. Suppose for the sake of contradiction that both L and L' are optimal choices. Now consider the choice $\frac{1}{2}L + \frac{1}{2}L'$. This choice gives the firm a profit of

$$\begin{aligned} & p_f g\left(\frac{1}{2}L_0 + \frac{1}{2}L'_0, \frac{1}{2}L_1 + \frac{1}{2}L'_1\right) - w_0\left(\frac{1}{2}L_0 + \frac{1}{2}L'_0\right) - w_1\left(\frac{1}{2}L_1 + \frac{1}{2}L'_1\right) \\ & > p_f \left(\frac{1}{2}g(L_0, L_1) + \frac{1}{2}g(L'_0, L'_1)\right) - w_0\left(\frac{1}{2}L_0 + \frac{1}{2}L'_0\right) - w_1\left(\frac{1}{2}L_1 + \frac{1}{2}L'_1\right) \\ & = \frac{1}{2}(p_f g(L_0, L_1) - w_0 L_0 - w_1 L_1) + \frac{1}{2}(p_f g(L'_0, L'_1) - w_0 L'_0 - w_1 L'_1) \\ & = p_f g(L_0, L_1) - w_0 L_0 - w_1 L_1. \end{aligned}$$

So this new choice gives a higher profit than L , which contradicts the assumption that L is optimal.

Part B

- (i) (easy) Let $f : X \rightarrow Y$ be a continuous function between two metric spaces (X, d_X) and (Y, d_Y) . Prove or disprove that $f(\partial A) = \partial f(A)$ for all sets $A \subseteq X$. *Note: ∂A denotes the set of boundary points of A , and $f(A) = \{f(a) : a \in A\}$.*

Answer. It is false. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ and the set $A = [-1, 1]$. Then $f(\partial A) = \{1\}$ and $\partial f(A) = \{0, 1\}$.

- (ii) (easy) Consider the metric space (\mathbb{R}, d) , where d is the discrete metric. Find a contraction $f : \mathbb{R} \rightarrow \mathbb{R}$ on this space.

Answer. Consider the function $f(x) = 0$. This function is a contraction of degree 0, since $d(f(x), f(y)) = 0 = 0d(x, y)$ for all $x, y \in \mathbb{R}$.

(iii) (easy) Consider the sequence $f_n : \mathbb{N} \rightarrow [0, 1]$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x < n, \\ 0 & \text{if } x \geq n. \end{cases}$$

Prove that this sequence is not convergent in $(B(\mathbb{N}), d_\infty)$.

Answer. $d_\infty(f_n, f_m) = 1$ for all $n \neq m$, since for all $n < m$, $f_n(n+1) = 0$ and $f_m(n+1) = 1$. This means f_n is not a Cauchy sequence, and hence is non-convergent.

(iv) (medium) Find a metric d such that the sequence f_n in the previous question converges to $f^*(x) = 1$.

Answer. Consider the metric

$$d(f, g) = \sup_{x \in \mathbb{N}} \frac{1}{x} |f(x) - g(x)|.$$

According to this metric,

$$d(f_n, f^*) = \frac{1}{n} |f_n(n) - f^*(n)| = \frac{1}{n}.$$

Since $d(f_n, f^*) \rightarrow 0$, we conclude that $f_n \rightarrow f^*$.

(v) (medium) Let $X = [0, 1)$. Find a metric d such that (X, d) is a compact metric space.

Answer. Consider the metric $d(x, y) = \min \{d_2(x, y), d_2(x-1, y), d_2(x+1, y)\}$, which corresponds to distances on a circle. For future reference, notice that $d(x, y) \leq d_2(x, y)$.

To see that (X, d) is compact, pick any sequence $x_n \in X$. Since $([0, 1], d_2)$ is compact (by the Bolzano-Weierstrass theorem), we know that x_n has a subsequence y_n converging to some $y^* \in [0, 1]$ according to the d_2 metric. If $y^* \neq 1$, then $y_n \rightarrow y^*$ in (X, d) , since $d(y_n, y^*) \leq d_2(y_n, y^*)$ and $d_2(y_n, y^*) \rightarrow 0$. If $y^* = 1$, then we claim that $y_n \rightarrow 0$. To see this, notice that $d_2(y_n, 1) = d_2(y_n - 1, 0) \geq d(y_n, 0)$ and $d_2(y_n, 1) \rightarrow 0$. In either case, y_n is a convergent subsequence of x_n inside (X, d) , so (X, d) is compact.

(vi) (medium) Suppose you are considering buying a house at market price p , which you value at v . But you don't want to buy if it has any (major) defects. You have taken a quick look already, and you think the probability of defects is q . You can pay inspectors c for conditionally independent reports about the house, which have type 1 and 2 errors of x and y . Each day, you choose whether to buy the house, to buy another report, or to give up. You discount days at rate β . You have a Bellman equation

$$\begin{aligned} V(q) = \max \{ & 0, qv - p, \\ & -c + [qx + (1-q)y]\beta V\left(\frac{qx}{qx + (1-q)y}\right) \\ & + [q(1-x) + (1-q)(1-y)]\beta V\left(\frac{q(1-x)}{q(1-x) + (1-q)(1-y)}\right) \}. \end{aligned}$$

Prove that the optimal policy involves giving up for low q , i.e. there exists $q_1 \in [0, 1]$ such that giving up is optimal for all $q \in [0, q_1]$.

Answer. It is possible to apply Blackwell's lemma to establish that the Bellman operator $F : CB[0, 1] \rightarrow CB[0, 1]$ defined by

$$F(V)(q) = \max \left\{ 0, qv - p, \right. \\ \left. -c + [qx + (1-q)y]\beta V \left(\frac{qx}{qx + (1-q)y} \right) \right. \\ \left. + [q(1-x) + (1-q)(1-y)]\beta V \left(\frac{q(1-x)}{q(1-x) + (1-q)(1-y)} \right) \right\}.$$

is a contraction inside the complete metric space $(CB[0, 1], d_\infty)$. So Banach's fixed point theorem establishes that there is a unique fixed point V^* among $CB[0, 1]$.

Moreover, $I = \{f \in CB[0, 1] : f \text{ is weakly increasing}\}$ is a closed and hence complete subspace of $CB[0, 1]$. Moreover, $F(I) \subseteq I$. So Banach's fixed point theorem implies that $V^* \in I$, i.e. V^* is weakly increasing.

Now, $V^*(0) = 0$. To see this, notice that $V^*(0)$ is the maximum of 0 , $-p$, and $-c + \beta V^*(0)$. If $V^*(0) > 0$, then we would have $V^*(0) = -c + \beta V^*(0)$, which is impossible. So $V^*(0) = 0$. This means that giving up is optimal if and only if $V^*(0) = 0$, which is true if and only if q is below some cut-off q_1 (since V^* is weakly increasing).

(vii) (hard) Define

$$f^n(x) = \begin{cases} f(f^{n-1}(x)) & \text{if } n \geq 1 \\ x & \text{if } n = 0. \end{cases}$$

Prove or disprove: if $f : [0, 1] \rightarrow [0, 1]$, $f(1) = 1$, and $f(x) > x$ for all $x < 1$, then $\lim_{n \rightarrow \infty} f^n(0) = 1$.

Answer. This is false. Consider the function

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{4} & \text{if } x < 0.5 \\ \frac{1}{4}x + \frac{3}{4} & \text{if } x \geq 0.5. \end{cases}$$

First, notice that $f(x) > x$ for all $x < 1$ and $f(1) = 1$.

Second, notice that if $x < 0.5$, then

$$f(f(x)) = \frac{1}{2} \left[\frac{1}{2}x + \frac{1}{4} \right] + \frac{1}{4} \\ = \frac{1}{2^2}x + \frac{1}{4} \left[\frac{1}{2^1} + 1 \right]$$

More generally,

$$f^n(x) = \frac{1}{2^n}x + \frac{1}{4} \left[\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^0} \right]$$

So we conclude that $\lim_{n \rightarrow \infty} f^n(0) = 0 + \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2}$, not 1.

(viii) (hard) Suppose (X, d_X) and (Y, d_Y) are metric spaces, where $K = X \cap Y$ is a compact set in both spaces. Suppose that $d_X(a, b) = d_Y(a, b)$ for all $a, b \in K$. Let $Z = X \cup Y$. Construct a metric d_Z on Z such that $d_Z(a, b) = d_X(a, b)$ for all $a, b \in X$, and $d_Z(a, b) = d_Y(a, b)$ for all $a, b \in Y$. *Hint: use the fact that d_X and d_Y are continuous. Note: a complete proof is long, with lots of cases to consider. You can get an almost perfect score for the “proofs” learning outcome by showing 1 or 2 cases well.*

Answer. Let $J = X \setminus K$ and let $L = Y \setminus K$. This means that $Z = J \cup K \cup L$, and every point in Z is in exactly one of these three sets.

Note that d_Z is already defined on all of Z^2 except $(J \times L) \cup (L \times J)$. Since we require d_Z to be symmetric with $d_Z(j, \ell) = d_Z(\ell, j)$, we only need to define d_Z on $J \times L$.

Pick any $j \in J$ and any $\ell \in L$. We define $d_Z(j, \ell) = \min_{k \in K} d_X(j, k) + d_Y(k, \ell)$. Consider the function $f : K \rightarrow \mathbb{R}$ between the metric space (K, d_X) and (\mathbb{R}, d_2) defined by $f(k) = d_X(j, k) + d_Y(k, \ell)$. Note that $(K, d_X) = (K, d_Y)$. Since $d_X(j, \cdot)$ and $d_Y(\cdot, \ell)$ are continuous, it follows that f is continuous. Since the domain K is compact, the extreme value theorem implies that the minimisation problem has a solution k^* . So $d_Z(j, \ell)$ is well-defined. Since $k^* \neq j$, we know that $d_Z(j, \ell) > 0$.

Thus $d_Z(z, z') = 0$ if and only if $z = z'$, and $d_Z(z, z') = d_Z(z', z)$. It remains to check the triangle inequality.

Pick any three points $z, z', z'' \in Z$. We need to show that $d_Z(z, z'') \leq d_Z(z, z') + d_Z(z', z'')$. There are 27 cases to consider, as each of z, z' , and z'' can be in either J, K , or L :

- $z, z', z'' \in X$ or $z, z', z'' \in Y$: the triangle inequality is inherited from d_X or d_Y , respectively. This accounts for 15 cases ($JJJ, JJK, JKJ, JKK, KJJ, KJK, KKJ, KKK, LLL, LLK, LKL, LKK, KLL, KLK, KKL$).
- $z \in J, z' \in K$ and $z'' \in L$: this accounts for 2 cases (JKL, LKJ). On the right side, $d_Z(z, z') = d_X(z, z')$ and $d_Z(z', z'') = d_Y(z', z'')$. On the left side, $d_Z(z, z'') = \inf_{k \in K} d_X(z, k) + d_Y(k, z'') \leq d_X(z, z') + d_Y(z', z'')$. So $d_Z(z, z'') \leq d_Z(z, z') + d_Z(z', z'')$ as required.
- $z \in J, z' \in L$ and $z'' \in L$: this accounts for 4 cases (JLL, LLJ, JLL, LJL). On the right side, $d_Z(z, z') = d_X(z, k^*) + d_Y(k^*, z')$ for some $k^* \in K$. On the left side,

$$\begin{aligned} d_Z(z, z'') &= \inf_{k \in K} d_X(z, k) + d_Y(k, z'') \\ &\leq d_X(z, k^*) + d_Y(k^*, z'') \\ &\leq d_X(z, k^*) + d_Y(k^*, z') + d_Y(z', z'') \\ &= d_Z(z, z') + d_Z(z', z''), \end{aligned}$$

as required.

- $z \in K, z' \in J$ and $z'' \in L$: this accounts for 4 cases (KJL, KLJ, JLK, LJK). On the right side, $d_Z(z', z'') = d_X(z', k^*) + d_Y(k^*, z'')$ for some $k^* \in K$.

By the triangle inequality for d_Y , we know $d_Y(z, z'') \leq d_Y(z, k^*) + d_Y(k^*, z'')$.
Combining, we deduce

$$\begin{aligned} d_Y(z, z'') &\leq d_Y(z, k^*) + d_Y(k^*, z'') \\ &= d_X(z, k^*) + d_Y(k^*, z'') \\ &\leq [d_X(z, z') + d_X(z', k^*)] + d_Y(k^*, z'') \\ &= d_X(z, z') + d_Z(z', z''), \end{aligned}$$

as required.

- $z \in J, z' \in L$ and $z'' \in J$: this accounts for 2 cases (JLJ, LJL). This case is trivial, because $d_Z(z, z'') = 0$.

42: Micro 1, May 2021

Write down a two-period model in which a pandemic strikes in the second period. In both periods, all households split their time between working, studying music online and/or offline (for leisure), and other leisure (e.g. watching free videos). A music company hires workers to supply piano lessons, and a restaurant hires workers to make meals. In the second period, two things change. First, music lessons move online, which is less fun for the student, and more work for the teacher. Second, people must eat restaurant meals at home, which is less fun. All workers can do both jobs. All households own an equal share of the firms.

- (i) Formulate a competitive model of the music education, labour, and restaurant meal markets.

Comment from Sean Ferguson. A common mistake was to define labour and non-lesson leisure separately without including any link between them, either as a definition or as a ‘time budget’ constraint on the households’ problem. Answers that did this would often list labour as a choice variable for households but leave it unconnected to the arguments of the utility function (which would prevent the households’ problem from having a solution, since a household could always increase its income costlessly by increasing its labour supply). Other answers did include the time constraint, but as a market clearing condition, which is incorrect - it is a constraint on what each household can choose to do, not a condition for equating supply and demand of a good.

Many otherwise good answers did not capture the change in household preferences in period 2 (meals and lessons less enjoyable), or the change in music lesson productivity (more work to produce a music lesson) or both. All that’s needed is some notation to distinguish between the utility and production functions in the two periods (i.e. a time subscript). A discount factor alone (with the same utility function in each period) was not sufficient to capture the change in preferences, because the reduction in enjoyment is described as applying to specifically meals and lessons, and not to the third-good, non-lesson leisure.

Some answers treated labour for the restaurant and for the music company as distinct goods with different wage variables. This is a valid approach, if a little more complicated (and if the two kinds of labour are perfect substitutes in the utility function, the wages in each period will have to turn out the same anyway, as long as both firms employ at least some labour). Students who did this and allowed households to choose how much of each type of labour to supply in each period generally did fine. Students who assumed that households could only work at one job and either had to choose a job or were only able to do one (presumably due to talent/qualifications etc.) struggled more. It is possible to write a good model along these lines, but it requires some extra care, particularly around the market clearing conditions - if only some households do a particular job, then only the labour supply of those households should be included in the market clearing condition for that labour market (you can sum over all households and use an indicator variable defined to be equal to 1 when they have/choose a given job, or indicate on your summation that you are summing over households with a particular

job). If households are assigned to one job or the other and cannot choose, then you cannot (at least without justification!) assume that they make the same choices in regard to non-labour variables - such a model has two types of household with different endowments (capacity to supply labour for different forms of production), and so the market clearing conditions can't rely on simply multiplying a particular choice by the total number of households.

Answer.

Households. Each local household $h \in H$ chooses working time ℓ_{ht} , studying time s_{ht} , and food consumption c_{ht} , which trade at prices w_t , p_t and q_t , respectively. This gives household h a utility of

$$\sum_{t=1}^2 u_t(s_{ht}, 1 - \ell_{ht} - s_{ht}, c_{ht}).$$

Each household receives dividends of $(\pi^r + \pi^s)/|H|$. The utility maximisation problem for household h is

$$\begin{aligned} \max_{\ell_{ht}, s_{ht}, c_{ht}} \quad & \sum_{t=1}^2 u_t(s_{ht}, 1 - \ell_{ht} - s_{ht}, c_{ht}) \\ \text{s.t.} \quad & \sum_{t=1}^2 (p_t s_{ht} + q_t c_{ht}) = \sum_{t=1}^2 w_t \ell_{ht} + \frac{\pi^r + \pi^s}{|H|} \end{aligned}$$

Restaurant. In period t , the restaurant hire L_t^r workers and produces $C_t = g_t^r(L_t^r)$ meals. Its profit function is

$$\pi^r(q_1, q_2, w_1, w_2) = \max_{L_1^r, L_2^r} q_1 g_1^r(L_1^r) + q_2 g_2^r(L_2^r) - w_1 L_1^r - w_2 L_2^r.$$

Music school. The music school hires L_t^s workers and produces $S_t = g_t^s(L_t^s)$ lessons. Its profit function is

$$\pi^s(p_1, p_2, w_1, w_2) = \max_{L_1^s, L_2^s} p_1 g_1^s(L_1^s) + p_2 g_2^s(L_2^s) - w_1 L_1^s - w_2 L_2^s.$$

Equilibrium. The prices (p_t, q_t, w_t) and quantities $(\ell_{hst}, s_{ht}, c_{ht}, L_t^r, L_t^s)$ form an equilibrium if the quantities are optimal choices in the problems above, and all

markets clear:

$$\begin{aligned} \sum_{h \in H} c_{h1} &= g_1^r(L_1^r) \\ \sum_{h \in H} c_{h2} &= g_2^r(L_2^r) \\ \sum_{h \in H} s_{h1} &= g_1^s(L_1^s) \\ \sum_{h \in H} s_{h2} &= g_2^s(L_2^s) \\ \sum_{h \in H} \ell_{h1} &= L_1^r + L_1^s \\ \sum_{h \in H} \ell_{h2} &= L_2^r + L_2^s. \end{aligned}$$

- (ii) Reformulate the households' problem using a Bellman equation in which the second period choices are buried inside a value function.

Comment from Sean Ferguson. Most answers to this question were correct. Most mistakes either included a saving variable in the second-period value function without adding it to the budget constraint of the Bellman equation (so that saving 'comes from nowhere') or separated the two periods into separate value functions without connecting them through a saving variable, which is not rewriting the households' problem but changing it by constraining them to spend all income from each period on consumption for that period.

Answer.

$$\begin{aligned} \max_{\ell_{h1}, s_{h1}, c_{h1}, m_h} \quad & u_t(s_{h1}, 1 - \ell_{h1} - s_{h1}, c_{h1}) + V_h(m_h) \\ \text{s.t.} \quad & p_1 s_{h1} + q_1 c_{h1} + m_h = w_1 \ell_{h1} + \frac{\pi^r + \pi^s}{|H|} \end{aligned}$$

where

$$\begin{aligned} V_h(m_h) = \max_{\ell_{h2}, s_{h2}, c_{h2}} \quad & u_t(s_{h2}, 1 - \ell_{h2} - s_{h2}, c_{h2}) \\ \text{s.t.} \quad & p_2 s_{h2} + q_2 c_{h2} = w_2 \ell_{h2} + m_h. \end{aligned}$$

- (iii) Assume that the relevant utility functions are strictly concave. Prove that in the Bellman equation you just wrote down, there is at most one optimal savings choice.

Comment from Sean Ferguson. Many answers asserted that the second-period value function was concave without giving an argument for it. It's generally important to show you understand the difference between the value function and the objective function - the question states that the utility function (i.e. the objective function of the second-period value function and a component of the objective function of the Bellman) is concave, but you need to argue from there to concavity of the objective of the Bellman.

Some answers made an argument for a unique equilibrium of the economy, usually incorporating an assertion that household problem has a unique solution (which is what the question asked for a proof of!). Questions about general aspects of the household and firm problems are not typically questions about equilibrium, because those problems are defined for any prices, not just equilibrium ones (the problems are the more general part of the model - any given equilibrium is one particular set of solutions to those problems for a particular set of prices, but the problems will also have many non-equilibrium solutions for other sets of prices that do not lead to market clearing).

Answer. Suppose the utility function u is strictly concave. By a theorem in the notes, the value function V is strictly concave, because the objective is strictly concave in the choice and state variables (quantities and savings) and the constraint is jointly quasiconcave in the choice and state variables (since it is linear in both). Therefore, the objective in the Bellman equation is strictly concave, and the menu is convex. So if there were two optimal choices, $(\ell_{h1}, s_{h1}, c_{h1}, m_h)$ and $(\ell'_{h1}, s'_{h1}, c'_{h1}, m'_h)$, then any (non-trivial) convex combination is feasible and gives strictly higher utility – a contradiction. So there is at most one optimal choice $(\ell_{h1}, s_{h1}, c_{h1}, m_h)$, and hence there is at most one possible choice of m_h .

- (iv) The government would like to encourage more people to work in the second period. To this end, it plans a lump sum tax on workers, which funds a subsidy to the firms in the second period. Would this policy lead to more work in the second period?

Comment from Sean Ferguson. Many students answered this correctly. One of the most common mistakes was to argue that the government could not achieve its goal because the first welfare theorem means that the original equilibrium is efficient. This means that the government cannot make everyone better off, but the question does not ask about that - there is no reason to think that people working more in the second period would make everyone better off! It is possible in general for lump sum taxes to change economic outcomes in these models (such as how much people work, how much of particular goods are produced, who consumes them, etc.), as long as the result is also efficient, bearing in mind that many different efficient outcomes exist for any economy, some better for some households and some better for others - what the first welfare theorem indicates is that you can't make an efficiency improvement that would be preferred by all the households. To change outcomes you need to make some households better off and some worse off - the point of this question is that moving money from households to firms doesn't do that, since the households own the firms and just get the money back.

Answer. No, the lump-sum transfers to the firms would trickle down to the owners, so nothing would change. Specifically, suppose a tax of T were levied on each household, and a bail-out of B^r were paid to the restaurant and B^s were paid to the music school, where $|H|T = B^r + B^s$. Then the firms' profits would be

$$\pi^r(q_1, q_2, w_1, w_2) + B^r$$

and

$$\pi^s(p_1, p_2, w_1, w_2) + B^s.$$

Thus, each household would receive an extra dividend of $(B^r + B^s)/|H|$, but pay an extra tax of T . But these two quantities are equal, so they cancel out.

- (v) Suppose that at market prices, all labour markets clear, all education markets clear, and that all households save none of their money for the second period. Prove that this implies that both restaurant meal markets clear. *Hint: think carefully about what it means to save for the future.*

Comment from Sean Ferguson. Many students got some credit for recognising the general logic that no saving essentially means Walras' Law can be applied separately to each time period. The best answers showed an understanding of why this is the case and how Walras' Law connects to the households' budget constraint.

Answer. This question is (intentionally) ambiguous – what does it mean to save money for the future, when all trade happens at the start? A good interpretation is: the period two choices, and the dividends from period two production are accounted for in the second period. According to this interpretation, it is helpful to split the firms up into two divisions, one for each time period, with profit functions $\pi_1^r(q_1, w_1)$, $\pi_2^r(q_2, w_2)$, $\pi_1^s(p_1, w_1)$, and $\pi_2^s(p_2, w_2)$. Household h 's savings are

$$a_h = w_1 \ell_{h1} + \frac{\pi_1^r(q_1, w_1)}{|H|} + \frac{\pi_1^s(q_1, w_1)}{|H|} - p_1 s_{h1} - q_1 c_{h1}.$$

So the question is asking us to assume that all household choose to save nothing, i.e. $a_h = 0$ for all h . Summing up savings across all households gives

Now, substituting in the profit functions gives:

$$\sum_{h \in H} a_h = \sum_{h \in H} w_1 \ell_{h1} - p_1 s_{h1} - q_1 c_{h1} + q_1 g_1^r(L_1^r) - w_1 L_1^r + p_1 g_1^s(L_1^s) - w_1 L_1^s.$$

We are told that the labour and education markets clear. After cancellation, we get

$$\sum_{h \in H} a_h = \sum_{h \in H} -q_1 c_{h1} + q_1 g_1^r(L_1^r).$$

Since we assumed the left side is zero, dividing both sides by q_1 gives

$$0 = - \sum_{h \in H} c_{h1} + g_1^r(L_1^r),$$

which means that the restaurant meal market in the first period clears.

This means we have established that all but one market (restaurant meals in the second period) clears. By Walras' law, all markets clear.

43: AME, December 2021

Part A

Suppose there are three years only. A fish farmer owns a fish farm, and is endowed with some adult trout (a fresh water fish). Each adult trout on the farm has children, which take one year to mature into adults. However, the children need to be cared for, otherwise many of them will die. Therefore, the fish farm hires workers to increase the fraction of child trout that survive. All households – both the farmer and the workers – choose how much labour to supply in years one and two, and how much trout to eat in all three years. A farm buys adult trout and hires workers in one year to make adult trout the following year.

- (i) Formulate a competitive model of the two labour markets and three fish markets.

Answer. Workers. There are n identical workers, who choose how much labour ℓ_t^w to supply at wage w_t in time $t \in \{1, 2\}$, and how much fish c_t^w to buy and eat at price p_t in time $t \in \{1, 2, 3\}$. This gives them utility

$$u(c_1^w, \ell_1^w) + \beta u(c_2^w, \ell_2^w) + \beta^2 u(c_3^w, 0),$$

where β is the discount rate. They solve the utility maximisation problem

$$\begin{aligned} \max_{\ell_1^w, \ell_2^w, c_1^w, c_2^w, c_3^w} & u(c_1^w, \ell_1^w) + \beta u(c_2^w, \ell_2^w) + \beta^2 u(c_3^w, 0) \\ \text{s.t.} & p_1 c_1^w + p_2 c_2^w + p_3 c_3^w = w_1 \ell_1^w + w_2 \ell_2^w. \end{aligned}$$

Farmer. The farmer owns the farm and is endowed with fish e_1^f , but is otherwise the same as the workers. His utility maximisation problem is

$$\begin{aligned} \max_{\ell_1^f, \ell_2^f, c_1^f, c_2^f, c_3^f} & u(c_1^f, \ell_1^f) + \beta u(c_2^f, \ell_2^f) + \beta^2 u(c_3^f, 0) \\ \text{s.t.} & p_1 c_1^f + p_2 c_2^f + p_3 c_3^f = p_1 e_1^f + w_1 \ell_1^f + w_2 \ell_2^f + \pi. \end{aligned}$$

Farm. In each year $t \in \{1, 2\}$, the farm chooses its fish stock S_t , and labour requirements L_t , and produces $f(S_t, L_t)$ the following year. So the firm has a net supply of fish of $C_1 = -S_1$, $C_2 = f(S_1, L_1) - S_2$ and $C_3 = f(S_2, L_2)$ in years 1, 2 and 3, respectively. The firm's profit maximisation problem is

$$\pi(p_1, p_2, p_3; w_1, w_2) = \max_{S_1, S_2, L_1, L_2} p_2 f(S_1, L_1) + p_3 f(S_2, L_2) - w_1 L_1 - w_2 L_2 - p_1 S_1 - p_2 S_2.$$

Equilibrium. Prices $(p_1, p_2, p_3, w_1, w_2)$ and quantities

$$(c_1^w, c_2^w, c_3^w, c_1^f, c_2^f, c_3^f, \ell_1^w, \ell_2^w, \ell_1^f, \ell_2^f, S_1, S_2, L_1, L_2, C_1, C_2, C_3)$$

forms an equilibrium if the choices solve the respective problems above, and all markets clear, i.e.

$$\begin{aligned} n c_1^w + c_1^f &= C_1 + e_1^f \\ n c_2^w + c_2^f &= C_2 \\ n c_3^w + c_3^f &= C_3 \\ n \ell_1^w + \ell_1^f &= L_1 \\ n \ell_2^w + \ell_2^f &= L_2. \end{aligned}$$

- (ii) Reformulate the farm's problem using a Bellman equation, in which the second and third year choices are buried inside a value function.

Answer. Let

$$V(p_2, p_3, w_2) = \max_{S_2, L_2} p_3 f(S_2, L_2) - w_2 L_2 - p_2 S_2,$$

Then

$$\pi(p_1, p_2, p_3; w_1, w_2) = \max_{S_1, L_1} p_2 f(S_1, L_1) - w_1 L_1 - p_1 S_1 + V(p_2, p_3, w_2).$$

- (iii) Prove that the firm's profit function is convex in the price of fish in the first year.

Answer. For each vector of choices (S_1, S_2, L_1, L_2) , the firm's profit is a linear function of all prices, and therefore of p_1 . Therefore, the profit function is the upper envelope of linear functions in p_1 . Since linear functions are convex, it follows that the profit function is the upper envelope of convex functions of p_1 . Therefore, π is convex in p_1 .

- (iv) What is the derivative of the firm's profit function with respect to the price of fish in the second year?

Answer. Let $(S_1^*, S_2^*, L_1^*, L_2^*)$ be the vector of optimal choices. By the envelope theorem,

$$\begin{aligned} & \frac{\partial \pi(p_1, p_2, p_3; w_1, w_2)}{\partial p_2} \\ &= \left[\frac{\partial}{\partial p_2} \{p_2 f(S_1, L_1) + p_3 f(S_2, L_2) - w_1 L_1 - w_2 L_2 - p_1 S_1 - p_2 S_2\} \right]_{(S_1, S_2, L_1, L_2) = (S_1^*, S_2^*, L_1^*, L_2^*)} \\ &= [f(S_1, L_1) - S_2]_{(S_1, S_2, L_1, L_2) = (S_1^*, S_2^*, L_1^*, L_2^*)} \\ &= f(S_1^*, L_1^*) - S_2^*. \end{aligned}$$

Part B

- (i) (Easy) Let (X, d) be a metric space. Find a counter-example to the false hypothesis, that every open ball $B_r(x)$ is connected.

Answer. Let $(X, d) = ([0, 2] \setminus \{1\}, d_2)$, and consider the ball $B_3(0)$. Notice that this ball equals the whole space X , which is disconnected.

- (ii) (Easy) Let (X, d) be a metric space. Prove that if every set $A \subseteq X$ is open, then every set $A \subseteq X$ is closed.

Answer. Pick any set $A \subseteq X$. Since the complement, $X \setminus A$, is a subset of X , it is an open set. Therefore A is closed.

- (iii) (Easy) Consider the metric spaces (X, d_X) and (Y, d_Y) . Suppose the function $f : X \rightarrow Y$ is bijective, and that f and f^{-1} are continuous. Prove that if $g : X \rightarrow X$ is discontinuous, then $h : X \rightarrow Y$ defined by $h(x) = f(g(x))$ is discontinuous.

Answer. I will prove the contrapositive of this statement, namely that if h is continuous, then g is continuous. To see this, notice that $g(x) = f^{-1}(h(x))$. Since g is the composition of two continuous functions, it is continuous.

- (iv) (Easy) A beer monopolist spends $c(q)$ to make q units of beer. He offers different prices to students and workers of p_s and p_w , respectively. Students and workers have inverse demand curves, $q_s, q_w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which are continuous, decreasing and $q_s(\bar{p}) = q_w(\bar{p}) = 0$ for some price $\bar{p} > 0$. Also assume that $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous. The monopolist's problem is

$$\begin{aligned} & \max_{p_s, p_w} p_s q_s(p_s) + p_w q_w(p_w) - c(q_s(p_s) + q_w(p_w)) \\ & \text{s.t. } p_s \geq 0 \text{ and } p_w \geq 0. \end{aligned}$$

Prove that there is an optimal solution, (p_s^*, p_w^*) to the monopolist's problem.

Answer. The monopolist would never want to set p_s or p_w above \bar{p} . Specifically, if the student price p_s were above \bar{p} , then the monopolist could lower the price to \bar{p} without changing profits (since $q_s = 0$ in either case), and the same logic applies to p_w . So the more tightly constrained problem,

$$\max_{(p_s, p_w) \in [0, \bar{p}]^2} p_s q_s(p_s) + p_w q_w(p_w) - c(q_s(p_s) + q_w(p_w)),$$

gives the same profit as the original one.

Since the domain is compact and the objective is continuous, the extreme value theorem implies that there exists an optimal choice, (p_s^*, p_w^*) .

- (v) (Medium) Consider the metric space (X, d) where $X = \mathbb{R}_+ \cup \{\infty\}$ and

$$d(x, y) = \begin{cases} \min\{1, |x - y|\} & \text{if } x, y \in \mathbb{R}_+, \\ 0 & \text{if } x = y = \infty, \\ 1 & \text{otherwise.} \end{cases}$$

Prove that (X, d) is not compact.

Answer. No. Consider the sequence $x_n = n$. Let y_n be any subsequence of x_n . I will show that y_n is not a Cauchy sequence, and is therefore not convergent. Hence (X, d) is not compact.

Notice that $d(x_n, x_m) > 1/2$ for all $n \neq m$. Since y_n is a subsequence, it follows that $d(y_n, y_m) > 1/2$ for all $n \neq m$. It follows that there is no N such that

$$d(y_n, y_m) < 1/2 \text{ for all } n, m > N.$$

So y_n is not a Cauchy sequence.

- (vi) (Medium) Find a counter-example to the following false claim: If (X, d) is a complete metric space and $f : X \rightarrow X$ is a contraction, then $f(X)$ is connected.

Answer. Let $X = \{1/n : n \in \mathbb{N}, n > 0\} \cup \{0\}$ and $d = d_1$. Let $f : X \rightarrow X$ be the function $f(x) = \frac{x}{2}$.

Notice that (X, d) is complete because X is a closed subset of (\mathbb{R}, d_1) .

Also notice that f is a contraction of degree $\frac{1}{2}$, since $d(f(x), f(y)) = \frac{1}{2}d(x, y)$.

Finally, notice that $(f(X), d)$ is a disconnected metric space, because $f(X) \cap [0, \frac{1}{4}]$ and $\{\frac{1}{2}\}$ are disjoint closed sets whose union is the whole space, $f(X)$.

- (vii) (Medium) Prove that (X, d) is connected if and only if every continuous function $f : X \rightarrow \{0, 1\}$ is constant.

Answer. Define the codomain as $(\{0, 1\}, d^*)$ where d^* is the discrete metric.

First, suppose (X, d) is connected and f is continuous. We need to prove that f is constant. Recall that range of continuous functions with connected domains is connected. But $\{0, 1\}$ is disconnected. So $f(X) = \{0\}$ or $f(X) = \{1\}$. So f is constant.

Conversely, suppose (X, d) is disconnected. We need to find an example of a continuous function $f : X \rightarrow \{0, 1\}$ that is not constant. Since (X, d) is disconnected, X is the disjoint union of two open sets A and B . Let $f(x) = I(x \in A)$. Notice that f is continuous since, because the pre-image of every (open) set in $(\{0, 1\}, d^*)$ is open. So f is continuous but not constant, as required.

- (viii) (Hard) A retired woman wakes up with a bank balance b and cash c in her wallet. She chooses how much of her cash to spend each day x , and how much leisure time to have, $\ell \leq 24$, which gives her utility $u(x, \ell)$. If she wants more cash, she has to walk 2 hours (round trip) to the bank. She doesn't withdraw all of her bank balance, because cash is exposed to inflation i , and banks pay interest on deposits to cancel out inflation. She discounts the future at rate β . Her value function solves the Bellman equation

$$\begin{aligned} V(b, c) = \sup_{x, b', c' \geq 0} & u(x, 24 - 2I(b' \neq b)) + \beta V(b', c') \\ \text{s.t. } & x + b' + c'(1 + i) = b + c, \end{aligned}$$

where $I(\cdot)$ is the indicator function, i.e. $I(b' \neq b)$ equals 1 if $b' \neq b$ and 0 otherwise. Suppose the utility function u is unbounded. Specify a suitable metric space for the domain of the Bellman operator, such that Banach's fixed point theorem can be applied to the Bellman operator. *Hint: divide the value functions by the maximum utility that can be achieved in one day.*

Answer. Let $X = \mathbb{R}_+^2$ be the state space, i.e. $(b, c) \in X$. Let $f(b, c) = \frac{u(b+c, 24)}{1-\beta}$, which is the discounted utility of consuming $b + c$ and relaxing every day. Let the distance metric d be

$$d(V, W) = \sup_{(b, c) \in X} \frac{|V(b, c) - W(b, c)|}{f(b, c)}.$$

Let $B^f(X) = \{V : X \rightarrow \mathbb{R} \text{ s.t. } \sup_{x \in X} V(x)/f(x) \leq 1\}$. Then a suitable space is $(B^f(X), d)$.

Let $G : B^f(X) \rightarrow B^f(X)$ be the Bellman operator, with

$$\begin{aligned} G(V)(b, c) = \sup_{x, b', c' \geq 0} & u(x, 24 - 2I(b' \neq b)) + \beta V(b', c') \\ \text{s.t. } & x + b' + c'(1 + i) = b + c. \end{aligned}$$

For a metric space to be suitable, the space must be complete and the Bellman operator G needs to be a contraction.

To see that the Bellman operator G is a self-map, notice that $f(b, c)$ is the value of consuming $b + c$ and relaxing every day. $B^f(X)$ consists of all value functions that are worse than f , i.e. $V(b, c) \leq f(b, c)$ for all $(b, c) \in X$. Now, if we relax the budget constraint so that (i) cash can be double-spent, i.e. spent and saved at the same time, and (ii) cash is no longer exposed to inflation so that bank trips are no longer necessary, then the Bellman operator becomes

$$\begin{aligned} H(V)(b, c) &= \sup_{x, b', c' \geq 0} u(x, 24) + \beta V(b', c') \\ &\quad \text{s.t. } x = b' + c' = b + c \\ &= u(b + c, 24) + \beta V(b, c) \\ &= (1 - \beta)f(b, c) + \beta V(b, c). \end{aligned}$$

Notice that $G(V)(b, c) \leq H(V)(b, c) \leq H(f)(b, c) = f(b, c)$ for all $(b, c) \in X$. The first inequality is because H involves a bigger menu than G . The second inequality is because f is the best value function in $B^f(X)$. The last equality is simple algebra:

$$H(f)(b, c) = (1 - \beta)f(b, c) + \beta f(b, c) = f(b, c).$$

We conclude that $G(V) \in B^f(X)$, so the Bellman operator G is a self-map.

Next, we prove that G is a contraction. Pick any $V, W \in B^f(X)$. Notice that for all (b, c) , any feasible choice of (b', c') has $f(b', c') \leq f(b, c)$. Therefore,

$$\begin{aligned} G(V)(b, c) &= \sup_{b', c' \geq 0} u(b + c - b' - c'(1 + i), 24 - 2I(b' \neq b)) + \beta V(b', c') \\ &= \sup_{b', c' \geq 0} u(b + c - b' - c'(1 + i), 24 - 2I(b' \neq b)) + \beta W(b', c') + \beta [V(b', c') - W(b', c')] \\ &\leq \sup_{b', c' \geq 0} u(b + c - b' - c'(1 + i), 24 - 2I(b' \neq b)) + \beta W(b', c') + \beta |V(b', c') - W(b', c')| \\ &\leq \sup_{b', c' \geq 0} u(b + c - b' - c'(1 + i), 24 - 2I(b' \neq b)) + \beta W(b', c') + \frac{f(b, c)}{f(b', c')} \beta |V(b', c') - W(b', c')| \\ &\leq \left[\sup_{b', c' \geq 0} u(b + c - b' - c'(1 + i), 24 - 2I(b' \neq b)) + \beta W(b', c') \right] \\ &\quad + \sup_{b', c' \geq 0} \frac{f(b, c)}{f(b', c')} \beta |V(b', c') - W(b', c')| \\ &= G(W)(b, c) + \beta f(b, c) d(V, W). \end{aligned}$$

Therefore

$$\frac{1}{f(b, c)} [G(V)(b, c) - G(W)(b, c)] \leq \beta d(V, W),$$

for all $(b, c) \in X$.

Similarly, by swapping the role of V and W above, we deduce

$$\frac{1}{f(b, c)}[G(W)(b, c) - G(V)(b, c)] \leq \beta d(V, W),$$

and therefore

$$\frac{1}{f(b, c)}|G(V)(b, c) - G(W)(b, c)| \leq \beta d(V, W),$$

for all $(b, c) \in X$. It follows that $d(G(V), G(W)) \leq \beta d(V, W)$. So G is a contraction of degree β .

Next, we prove that $(B^f(X), d)$ is complete. Suppose that $V_n \in B^f(X)$ is a Cauchy sequence. Let $W_n(x) = V_n(x)/f(x)$. By construction, $W_n \in B(X)$. Moreover, $d_\infty(W_n, W_m) = d(V_n, V_m)$, so W_n is a Cauchy sequence inside $(B(X), d_\infty)$. Since $(B(X), d_\infty)$ is complete, we deduce that W_n is convergent. Let W^* be the limit. It follows that V_n converges to $V^*(x) = W^*(x)f(x)$. So $(B^f(X), d)$ is complete.

44: Micro 1, December 2021

Consider the economy of two nearby and identical towns, Byron Bay and Casino. They are near enough to share a hospital, and households are indifferent between travelling to a hospital in either town. However, the towns are too far apart for workers to commute. A hospital requires many workers before it treats its first patient. Therefore, assume it is inefficient for both towns to operate their own hospital. On the other hand, both towns have a resort, which are perfect substitutes. Workers in each town supply labour to one of the local businesses, and consume holidays and treatments. Workers own equal shares in the local businesses. Hospitals and resorts only use labour to supply treatments and holidays.

- (i) Construct a competitive model of Byron Bay and Casino. *Hint: assume that there are two hospitals, but accommodate the possibility that the hospitals are inactive, i.e. hire no workers.*

Answer.

Households. There are n households in each town. Households in town $t \in \{B, C\}$ choose how much labour to supply, ℓ_t , how many holidays to take h_t , and how much medical treatment m_t to take, at prices w_t , p and q respectively, to maximise utility $u(1 - \ell_t, h_t, m_t)$. Each household receives $\frac{\pi_t^H}{n} + \frac{\pi_t^R}{n}$ in dividends from the local hospital and local resort. Their utility maximisation problem is

$$\begin{aligned} \max_{\ell_t, h_t, m_t} & u(1 - \ell_t, h_t, m_t) \\ \text{s.t.} & ph_t + qm_t = w_t\ell_t + \frac{\pi_t^H}{n} + \frac{\pi_t^R}{n} \end{aligned}$$

Hospitals. The hospital in town t chooses how much labour to hire L_t^H , and produces $f(L_t^H)$ treatments. We assume that $f(L_t^H) = 0$ for small values of L_t^H , so f is not concave. The hospital's profit function is

$$\pi_t^H(q; w_t) = \max_{L_t^H} qf(L_t^H) - w_tL_t^H.$$

Resorts. The resort in town t chooses how much labour to hire L_t^R , and produces $g(L_t^R)$ holidays. The resort's profit function is

$$\pi_t^R(p; w_t) = \max_{L_t^R} pg(L_t^R) - w_tL_t^R.$$

Equilibrium. Price (p, q, w_B, w_C) and quantities $(\ell_t, h_t, m_t, L_t^H, L_t^R)_{t \in \{B, C\}}$ form an equilibrium if the quantities solve the respective problems above and all markets clear:

$$\begin{aligned} n\ell_B &= L_B^H + L_B^R \\ n\ell_C &= L_C^H + L_C^R \\ nm_B + nm_C &= f(L_B^H) + f(L_C^H) \\ nh_B + nh_C &= g(L_B^R) + g(L_C^R). \end{aligned}$$

- (ii) Suppose the resorts merge into a single firm. Write down the merged firm's profit function, with and without a Bellman equation.

Answer. The merged firm's profit function is

$$\pi^R(p; w_B, w_C) = \max_{L_B^R, L_C^R} pg(L_B^R) + pg(L_C^R) - w_B L_B^R - w_C L_C^R.$$

It is related to the divisions' profit functions via the Bellman equation

$$\pi^R(p; w_B, w_C) = \pi_B^R(p; w_B) + \pi_C^R(p; w_C).$$

- (iii) Is there a lump-sum transfer scheme that implements perfect equality?

Answer. Yes. The Pareto frontier includes a point with perfect equality. (For Mathematical Microeconomics 1 students: this is because the utility possibility set is connected, so the intermediate value theorem applies.) So the second welfare theorem implies that there exist lump sum transfers to implement this egalitarian allocation.

- (iv) Suppose that all markets clear except the holiday markets in the two cities. Does this imply that both holiday markets clear? Does your answer depend on whether you model these two markets as a single market? Explain.

Answer. In my formulation, there is a single holiday market. If all other markets clear, then Walras law implies that the holiday market also clears.

But what if I had formulated the model with separate holiday markets in each town? Since the question states that the resorts are perfect substitutes, only holidays at the cheaper resort would be purchased. So in principle, there could be excess supply in one resort (completely empty), and excess demand in the other.

In other words, if we assume that the resorts have the same price, then all markets must clear. But not if we allow the resorts to have different prices.

- (v) Consider an equilibrium in which a hospital opens in Byron Bay only. Prove that the residents of Byron Bay work more than those of Casino.

Answer. The (inactive) Casino hospital and the Byron Bay hospital solve the same optimisation problem, apart from the possibility that they face different wages. Since only the Byron Bay hospital operates, it follows that either (i) $w_B < w_C$ or (ii) $w_B = w_C$ and both hospitals make zero profit, i.e. they are both indifferent between operating or not.

I now rule out this second possibility. If $w_B = w_C$, then the resorts would share the same optimisation problem, so $L_B^R = L_C^R$ and they make the same profits $\pi_B^R = \pi_C^R$. But the Byron Bay households also work in the hospital, so

$$n\ell_B = L_B^R + L_B^H = L_C^R + L_B^H > L_C^R = n\ell_C,$$

and hence $\ell_B > \ell_C$. On the other hand, the Byron Bay households have identical budget constraints to the Casino households – same prices, and same dividends.

This would imply $\ell_B = \ell_C$, a contradiction. So the premise was false, and I deduce that $w_B < w_C$.

Now consider the first-order conditions for the resorts, which are

$$w_t = pg'(L_t^R).$$

Since we already established that $w_B < w_C$, it follows that $g'(L_B^R) < g'(L_C^R)$, and therefore $L_B^R > L_C^R$. By the market clearing conditions,

$$n\ell_B = L_B^R + L_B^H > L_B^R > L_C^R = n\ell_C.$$

So we deduce that households in Byron Bay work harder, i.e. $\ell_B > \ell_C$.

45: AME, May 2022

Part A

A computer processor is faster if it has fewer defects, because the defective components must be disabled. For example, it might have fewer arithmetic units or less cache memory.

A processor factory (“fab”) hires workers for two tasks: production and quality control. The fab sells two types of processor: fully functional (fast) and defective (slow). Workers choose what processors to buy, allocate their time between work and leisure, own an equal share of the fab, and are all identical.

- (i) Formulate a competitive model of the labour market and the two processor markets.

Answer. Workers. There are n identical workers, who choose how much labour ℓ to supply at wage w , and how many processors x_s of speed $s \in \{g, b\}$ to buy at price p_s . This gives them a utility of $u(x_g, x_b, \ell)$. They receive a dividend of π/n , so their utility maximisation problem is

$$\begin{aligned} \max_{\ell, x_g, x_b} & u(x_g, x_b, \ell) \\ \text{s.t.} & p_g x_g + p_b x_b = w\ell + \frac{\pi}{n}. \end{aligned}$$

Fab. The fab hires L_x worker for production and L_q workers for quality control, and produces $X_g = f(L_x, L_q)$ fast processors and $X_b = g(L_x, L_q)$ slow processors. Its profit maximisation problem is

$$\pi(p_g, p_b; w) = \max_{L_x, L_q} p_g f(L_x, L_q) + p_b g(L_x, L_q) - w(L_x + L_q).$$

Equilibrium. Prices (p_g, p_b, w) and quantities

$$(x_g, x_b, \ell, X_g, X_b, L_x, L_q)$$

forms an equilibrium if the choices solve the respective problems above, and all markets clear, i.e.

$$\begin{aligned} nx_g &= X_g \\ nx_b &= X_b \\ n\ell &= L_x + L_q. \end{aligned}$$

- (ii) Reformulate the fab’s profit maximisation problem using a Bellman equation in which the firm’s choice of how to allocate its labour force across the two tasks is buried inside a value function.

Answer. Let $V(p_g, p_b; L)$ be the value of having L workers, defined by

$$\begin{aligned} V(p_g, p_b; L) &= \max_{L_x, L_q} p_g f(L_x, L_q) + p_b g(L_x, L_q) \\ \text{s.t.} & L_x + L_q = L. \end{aligned}$$

Then

$$\pi(p_g, p_b; w) = \max_L V(p_g, p_b; L) - wL.$$

(iii) Prove that if wages increase, then the firm hires fewer workers.

Answer. Recall

$$\pi(p_g, p_b; w) = \max_{L_x, L_q} p_g f(L_x, L_q) + p_b g(L_x, L_q) - w(L_x + L_q).$$

Let $(L_x^*, L_q^*) = (L_x(p_g, p_b; w), L_q(p_g, p_b; w))$ be an optimal labour demand choice. By the envelope theorem,

$$\begin{aligned} \frac{\partial \pi(p_g, p_b; w)}{\partial w} &= \left[\frac{\partial}{\partial w} \{p_g f(L_x, L_q) + p_b g(L_x, L_q) - w(L_x + L_q)\} \right]_{(L_x, L_q) = (L_x^*, L_q^*)} \\ &= [-(L_x + L_q)]_{(L_x, L_q) = (L_x^*, L_q^*)} \\ &= -L_x^* - L_q^* \\ &= -L_x(p_g, p_b; w) - L_q(p_g, p_b; w). \end{aligned}$$

The firm's objective is affine (and hence convex) in w . Thus, the profit function, which is the upper envelope of these functions (one function for each (L_x, L_q)) is convex. Therefore, the left side of the envelope equation,

$$\frac{\partial \pi(p_g, p_b; w)}{\partial w},$$

is increasing in w . So the right side is also increasing, so we conclude that labour demand, $L_x(p_g, p_b; w) + L_q(p_g, p_b; w)$ is decreasing in wages.

Part B

(i) (Easy) Recall that $CB(\mathbb{R})$ is the set of continuous and bounded functions whose domain and co-domain is (\mathbb{R}, d_2) . Let $X = \{f \in CB(\mathbb{R}) : f(0) = 0 \text{ and } f(x) \geq 0\}$. Prove that (X, d_∞) is a complete metric space.

Answer. Recall that $(CB(\mathbb{R}), d_\infty)$ is complete. Thus it suffices to show that X is a closed subset, because closed subsets of complete metric spaces are complete.

Let $f_n \in X$ be a convergent sequence with $f_n \rightarrow f^*$. To show that X is closed, we need to show that $f^* \in X$. Since $f_n \in X$, we know that $f_n(0) = 0$. Now,

$$|f_n(0) - f^*(0)| \leq \sup_{a \in \mathbb{R}} |f_n(a) - f^*(a)| = d_\infty(f_n, f^*).$$

Since $f_n \rightarrow f^*$, we know that right side converges to zero. It follows that the left side converges to zero, so $f^*(0) = 0$.

Similarly, pick any $a \in \mathbb{R}$. Since $f_n \in X$, we know that $f_n(a) \geq 0$. As before,

$$|f_n(a) - f^*(a)| \leq d_\infty(f_n, f^*).$$

Since the right side converges to zero, so does the left side, and hence $f_n(a) \rightarrow f^*(a)$. Now, since $f_n(a) \in \mathbb{R}_+$, and \mathbb{R}_+ is a closed set, we deduce that $f^*(a) \in \mathbb{R}_+$. So $f^*(a) \geq 0$.

We conclude that $f^* \in X$, as required.

- (ii) (Easy) Let (X, d_X) and (Y, d_Y) be metric space. Consider (Z, d_Z) , where $Z = X \times Y$ and $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$. Prove that if U is an open set inside (Z, d_Z) , then $V = \{x \in X : (x, y) \in U\}$ is an open set inside (X, d_X) .

Answer. Pick any point $x \in V$. We want to find a radius $r > 0$ such that $B_r(x) \subseteq V$.

Since $x \in V$, there exists some $y \in Y$ such that $(x, y) \in U$. Since U is open, there exists some radius $r > 0$ such that $B_r(x, y) \subseteq U$. Thus, if $x' \in B_r(x)$, then $d_X(x, x') < r$ and hence $d_Z(x, y; x', y) = d_X(x, x') < r$. We deduce that $(x', y) \in B_r(x, y) \subseteq U$ and conclude that $x' \in V$.

- (iii) (Easy) Consider a metric space (X, d) . Prove that if $U \subseteq V \subseteq X$, then the interior of U is contained in the interior of V , i.e. $\text{int}(U) \subseteq \text{int}(V)$.

Answer. Suppose $u \in \text{int}(U)$. We need to prove that $u \in \text{int}(V)$.

Since $u \in \text{int}(U)$, there exists a radius $r > 0$ such that $B_r(u) \subseteq U$. Since $U \subseteq V$, it follows that $B_r(u) \subseteq V$. We conclude that $u \in \text{int}(V)$, as required.

- (iv) (Easy) Prove that a metric space (X, d) is connected if and only if there does not exist two sets A and B such that $X = A \cup B$ and their closures are disjoint, i.e. $\text{cl}(A) \cap \text{cl}(B) = \emptyset$.

Answer. I prove the contrapositives of the two statements.

Suppose (X, d) is disconnected, i.e. there are two disjoint open sets A and B such that $X = A \cup B$. Since $A = X \setminus B$ is the complement of an open set, A is closed. Similarly B is closed. Therefore $A = \text{cl}(A)$ and $B = \text{cl}(B)$. Since A and B are disjoint, we deduce that $\text{cl}(A) \cap \text{cl}(B) = \emptyset$.

Conversely, suppose A and B have disjoint closures and $A \cup B = X$. Since $A \subseteq \text{cl}(A)$ and $B \subseteq \text{cl}(B)$, we know that $\text{cl}(A) \cup \text{cl}(B) = X$. Since the closures are disjoint, it follows that $\text{cl}(A)$ is the complement of $\text{cl}(B)$, so both are open. We conclude that (X, d) is disconnected.

- (v) (Medium) Let $e : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function where $e(-1) = -1$ and $e(1) = 1$. Consider the following optimisation problem,

$$\begin{aligned} \max_{\bar{u} \in \mathbb{R}} \bar{u} \\ \text{s.t. } e(\bar{u}) = 0. \end{aligned}$$

(This is a simplified version of the Bellman equation connecting the indirect utility function and the expenditure function, which are not examinable.) Prove that there exists a solution, \bar{u}^* .

Answer. The problem can be reformulated as follows

$$\max_{\bar{u} \in U} \bar{u},$$

where $U = e^{-1}(\{0\})$.

Since e is continuous, $e(1) = 1$ and $e(-1) = -1$, the intermediate value theorem implies that U is non-empty.

Since e is continuous and $\{0\}$ is closed, U is closed inside $([-1, 1], d_2)$. Since $([-1, 1], d_2)$ is a compact metric space, and U is a closed subset, we deduce that U is compact.

Thus the reformulated problem has a continuous objective and a non-empty and compact choice set. So the extreme value theorem implies it has a solution \bar{u}^* .

- (vi) (Medium) Let U be a connected set inside the metric space (X, d) . Prove that the closure of U is connected.

Answer. Before proving this, I will prove a useful fact: if A is closed in (X, d) , then $A' = A \cap U$ is closed inside (U, d) . To see this, notice that if $a_n \in A'$ converges to a^* inside (U, d) , then it also converges inside (X, d) . Since A is closed, $a^* \in A$. Since $a^* \in U$, we conclude $a^* \in A \cap U = A'$.

Now, back to the question at hand. I will prove the contrapositive, that if $\text{cl}(U)$ is disconnected, then U is disconnected.

Since $\text{cl}(U)$ is disconnected, there exist two disjoint closed sets A, B such that $A \cup B = \text{cl}(U)$. Let $A' = A \cap U$ and let $B' = B \cap U$, which are both closed inside the metric space (U, d) – see the useful fact above. Moreover, $A' \cup B' = U$, since $A' \cup B' = (A \cap U) \cup (B \cap U) = (A \cup B) \cap U = \text{cl}(U) \cap U = U$.

I conclude that (U, d) is disconnected.

- (vii) (Medium) Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a non-empty non-closed set A such that f is discontinuous at points inside of A and continuous elsewhere. Hint: consider using the indicator function $g(x) = I(x \in \mathbb{Q})$ as a building block.

Answer. Let $g(x) = I(x \in \mathbb{Q})$, which is discontinuous everywhere. Let $h(x) = \max\{0, 1 - x^2\}$, which is continuous everywhere. Let $f(x) = g(x)h(x)$, which I will show is discontinuous on $A = (-1, 1)$, which is not closed set, and continuous elsewhere.

Pick any $x^* \in A$. Since A is an open set, there is an open ball $B_r(x^*)$ such that $f(x) = (1 - x^2)g(x)$ for all $x \in B_r(x^*)$. If x^* is rational, then $f(x^*) = 1 - (x^*)^2$. In this case, we can pick any irrational sequence $x_n \rightarrow x^*$ inside A . Then $f(x_n) = 0$, so $f(x_n) \rightarrow 0 \neq f(x^*)$, and we conclude f is discontinuous at x^* .

If x^* is irrational, then $f(x^*) = 0$. In this case, we can pick any rational sequence $x_n \rightarrow x^*$ inside A . Then $f(x_n) = x_n^2 - 1$ so $f(x_n) \rightarrow 1 - (x^*)^2 \neq f(x^*)$, and we conclude f is discontinuous at x^* .

Finally, pick any $x^* \notin A$. Let x_n be any sequence converging to x^* . Then $0 \leq f(x_n) \leq h(x_n)$. Since $x^* \notin A$, $h(x^*) = 0$, and hence $h(x_n) \rightarrow 0$. We deduce that $f(x_n) \rightarrow 0$, by the squeeze theorem. Therefore, f is continuous at x^* .

To conclude: f is discontinuous on A and continuous outside of A .

- (viii) (Hard) Prove that there is no continuous injective function $f : [0, 1]^2 \rightarrow [0, 1]$, where both spaces use d_2 .

Answer. Consider the sets

$$U = \{(x, y) \in [0, 1]^2 \text{ s.t. either } x = 0 \text{ or } y = 0\},$$
$$V = \{(x, y) \in [0, 1]^2 \text{ s.t. either } x = 1 \text{ or } y = 1\}.$$

(Geometrically speaking, U consists of the bottom and left sides of the square, and V consists of the top and right sides of the square.) Notice that both sets are connected, and that they touch each other at the bottom-left and top-right corners, i.e. $U \cap V = \{(0, 0), (1, 1)\}$.

Pick any continuous function $f : [0, 1]^2 \rightarrow [0, 1]$. Let $a = f(0, 0)$, let $b = f(1, 1)$, and let c be a point in the middle, such as $\frac{1}{2}a + \frac{1}{2}b$. By the intermediate value theorem there exists a point $u \in U$ (on the bottom or left side) such that $f(u) = c$. Similarly, there is a point $v \in V$ (on the top or right side) such that $f(v) = c$. Since c is distinct from a and b , we deduce that u and v are not where the two sets touch at the bottom-left or top-right corners, i.e. $u, v \notin U \cap V$. So $u \neq v$.

Therefore $f(u) = c$ and $f(v) = c$ for distinct u and v , so f is not injective.

46: Micro 1, May 2022

A natural services company hires workers to do three tasks: field work, desk work and cleaning. Workers are identical. They prefer doing a mix of field work and desk work compared to specialising in one or the other, and they prefer either to cleaning. Workers supply the three types of labour and hire the company to maintain biodiverse habitats around their homes.

- (i) Construct a competitive model of the labour and natural services market. *Note: You do not need to model the details of labour preferences specified above, but your model must be general enough to accommodate them.*

Answer.

Households. There are n identical households. Each household supplies f hours of field work, d hours of desk work and c hours of cleaning at wages w_f , w_d and w_c , respectively. Each household hires the company to maintain b square metres of biodiverse habitats around their homes at price p . This gives them a utility of $u(b, f, d, c)$. They receive dividends of $\pi(p; w_f, w_d, w_c)/n$, so their utility maximisation problem is

$$\begin{aligned} \max_{b, f, d, c} u(b, f, d, c) \\ \text{s.t. } pb = w_f f + w_d d + w_c c + \frac{\pi}{n}. \end{aligned}$$

Nature company. The nature company hires F , D , C hours of field work, desk work and cleaning, respectively, and produces $B = g(F, D, C)$ square metres of biodiverse habitat. Its profit function is

$$\pi(p; w_f, w_d, w_c) = \max_{F, D, C} pg(F, D, C) - w_f F - w_d D - w_c C.$$

Equilibrium. Price (p, w_f, w_d, w_c) and quantities (b, f, d, c, B, F, D, C) form an equilibrium if the quantities solve the respective problems above and all markets clear:

$$\begin{aligned} nb &= B \\ nf &= F \\ nd &= D \\ nc &= C. \end{aligned}$$

- (ii) Write down a Bellman equation in which the company chooses output, and its labour choices are buried inside a value function.

Answer. Consider the cost function

$$\begin{aligned} H(B; w_f, w_d, w_c) &= \min_{F, D, C} w_f F + w_d D + w_c C \\ \text{s.t. } g(F, D, C) &= B. \end{aligned}$$

Then the nature company's profit function can be written as

$$\pi(p; w_f, w_d, w_c) = \max_B pB - H(B; w_f, w_d, w_c).$$

- (iii) Prove that if the cleaning wage increases, then the firm demands fewer hours of cleaning.

Answer. Let (F^*, D^*, C^*) be optimal choices for the prices $(p; w_f, w_d, w_c)$. By the envelope theorem,

$$\begin{aligned} & \frac{\partial \pi(p; w_f, w_d, w_c)}{\partial w_c} \\ &= \left[\frac{\partial}{\partial w_c} (pg(F, D, C) - w_f F - w_d D - w_c C) \right]_{(F, D, C) = (F^*, D^*, C^*)} \\ &= -C^*. \end{aligned}$$

Since this holds for all prices $(p; w_f, w_d, w_c)$, we deduce that

$$\frac{\partial \pi(p; w_f, w_d, w_c)}{\partial w_c} = -C(p; w_f, w_d, w_c).$$

The profit function π is the upper envelope of functions that are linear in prices, with one function per choice of inputs (F, D, C) . Since linear functions are convex, it follows that π is the upper envelope of convex functions. So π is convex.

Since π is convex, the left side of the envelope equation is increasing in w_c . Thus the right side is also increasing. I conclude that the factor demand function C is decreasing in w_c .

- (iv) Suppose the utility functions and production function are strictly increasing and strictly concave. Prove there is at most one equilibrium.

Answer. Since households have strictly concave utility functions, they have a uniquely optimal choice (given prices), so all equilibria are symmetric.

Now, suppose for the sake of contradiction that there were two symmetric equilibria with household quantities (b', f', d', c') and (b'', f'', d'', c'') . (The firm quantities can be calculated as $F' = nf'$ and so on.) By the first welfare theorem, both are efficient, and therefore give the same utility to the households.

Let $(f^*, d^*, c^*) = \frac{1}{2}(f', d', c') + \frac{1}{2}(f'', d'', c'')$. Since the production function is strictly concave, $g(f^*, d^*, c^*) > \frac{1}{2}g(f', d', c') + \frac{1}{2}g(f'', d'', c'')$. Thus, $b^* > \frac{1}{2}b' + \frac{1}{2}b''$.

I deduce that

$$\begin{aligned} u(b^*, f^*, d^*, c^*) &> u\left(\frac{1}{2}b' + \frac{1}{2}b'', f^*, d^*, c^*\right) \\ &= u\left[\frac{1}{2}(b', f', d', c') + \frac{1}{2}(b'', f'', d'', c'')\right] \\ &> \frac{1}{2}u(b', f', d', c') + \frac{1}{2}u(b'', f'', d'', c'') \\ &= u(b', f', d', c'), \end{aligned}$$

where the two strict inequalities use the fact that u is strictly increasing and strictly concave, respectively. Therefore, the household allocation (b^*, f^*, d^*, c^*) is feasible and Pareto dominates (b', f', d', c') . Therefore, (b', f', d', c') is an inefficient equilibrium allocation, violating the first welfare theorem. I conclude that the premise that there are two equilibria is false.

- (v) The government worries that cleaning is dangerous, so it proposes banning half the population from cleaning work. (For example, in some parts of India, the Brahmin caste is de facto banned from some types of cleaning.) Assume that in both cases, with and without the ban, there is a unique equilibrium. Prove that
- (a) the prices in the cleaning ban equilibrium are different from the original equilibrium, and
 - (b) the people banned from cleaning are made worse off.

Answer. Without loss of generality, in all price vectors below, I will assume that the price of biodiverse habitat is 1.

In the absence of the ban, let $(b^*, f^*, d^*, c^*, F^*, D^*, C^*)$ be the equilibrium quantities, and $(1, w_f^*, w_d^*, w_c^*)$ be the equilibrium prices. Suppose that $m = n/2$ households are allowed to clean, and m households are not. Then let $(b', f', d', 0)$ and (b'', f'', d'', c'') be the equilibrium quantities for the constrained and the unconstrained households, let (F', D', C') be the firm's equilibrium quantities and $(1, w'_f, w'_d, w'_c)$ be the corresponding equilibrium prices.

First I rule out $(w_f^*, w_d^*, w_c^*) = (w'_f, w'_d, w'_c)$. If the prices were the same, then the firm would make the same choices, i.e. $(F^*, D^*, C^*) = (F', D', C')$. So the unconstrained households' budget constraint would be the same, so their choices would also be unchanged, i.e. $(b'', f'', d'', c'') = (b^*, f^*, d^*, c^*)$. Thus, the aggregate cleaning hours supplied would be $0m + c''m = c^*m$, and the aggregate cleaning hours demanded would be $C' = C^* = c^*n$. Thus, the market clearing condition for cleaning is violated, i.e. $0m + c''m = c^*m \neq c^*n = C'$. I conclude that $(w_f^*, w_d^*, w_c^*) \neq (w'_f, w'_d, w'_c)$.

Next, notice that the constrained households are worse off than the unconstrained ones, i.e. $u(b', f', d', 0) < u(b'', f'', d'', c'')$. This is true because their optimisation problems are the same apart from the cleaning choices that were taken off the menu.

Also, notice that the constrained households are worse off than under the original equilibrium, i.e. $u(b', f', d', 0) < u(b^*, f^*, d^*, c^*)$. If this were not the case, then we would have both households better off in the constrained equilibrium with $u(b^*, f^*, d^*, c^*) \leq u(b', f', d', 0) < u(b'', f'', d'', c'')$. This means the original equilibrium was inefficient. But this contradicts the first welfare theorem.

47: Skipped.

48: AME, December 2022

Part A

Egypt and Sudan both depend on water from the Nile river. Since the Nile flows through Sudan first, Sudanese households are endowed with all of the water. A firm in each country buys wholesale water and hires local labour, and sells food internationally and retail water locally. Each household supplies labour and wholesale water (Sudan only), and buys food and local retail water. Each firm is owned by the local households.

- (i) Formulate a competitive equilibrium model of the international food and wholesale water markets, and the domestic water and labour markets.

Answer. Households. There are two countries, $c \in \{E, S\}$. Country c has a set H_c of identical households. Each household is endowed with 1 unit of time and a_c units of wholesale water, which it sells inelastically at prices w_c and p^a , respectively. (We assume that $a_E = 0$.) It also receives dividends $\pi_c/|H_c|$ from the local firm. It then buys x_c units of food at price p^x , b_c units of retail water at price p_c^b . The household's utility is $u(x_c, b_c)$, and its utility maximisation problem is

$$\begin{aligned} \max_{x_c, b_c} u(x_c, b_c) \\ \text{s.t. } p^x x_c + p_c^b b_c = w_c + p^a a_c + \pi_c/|H_c| \end{aligned}$$

Firms. The firm in country c buys A_c^x wholesale water and L_c^x labour for producing $f(A_c^x, L_c^x)$ units of food. It buys A_c^b wholesale water and L_c^b labour for producing $g(A_c^b, L_c^b)$ units of retail water. Its profit maximisation problem is

$$\pi_c(p^x, p_c^b; w_c, p^a) = \max_{A_c^x, L_c^x, A_c^b, L_c^b} p^x f(A_c^x, L_c^x) + p_c^b g(A_c^b, L_c^b) - p^a (A_c^x + A_c^b) - w_c (L_c^x + L_c^b).$$

Equilibrium. Prices $(p^x, p_E^b, p_S^b, p^a, w_E, w_S)$ and quantities $(x_c, b_c, A_c^x, L_c^x, A_c^b, L_c^b)_{c \in \{E, S\}}$ form an equilibrium if the quantities solve the respective problem above, and all markets clear:

$$\begin{aligned} |H_S| &= L_S^x + L_S^b \\ |H_E| &= L_E^x + L_E^b \\ |H_S|a_S &= A_S^x + A_S^b + A_E^x + A_E^b \\ |H_S|x_S + |H_E|x_E &= f(A_S^x, L_S^x) + f(A_E^x, L_E^x) \\ |H_S|b_S &= g(A_S^b, L_S^b) \\ |H_E|b_E &= g(A_E^b, L_E^b) \end{aligned}$$

- (ii) Reformulate the Egyptian firm's profit maximisation problem with a Bellman equation in which all choices except water demand are buried inside a value function.

Answer.

$$\pi_E(p^x, p_E^b; w_E, p^a) = \max_{A_E} V(A_E; p^x, p_E^b, w_E) - p^a A_E$$

where

$$V(A_E; p^x, p_E^b, w_E) = \max_{A_E^x, L_E^x, A_E^b, L_E^b} p^x f(A_E^x, L_E^x) + p_E^b g(A_E^b, L_E^b) - w_E(L_E^x + L_E^b)$$

$$\text{s.t. } A_E^x + A_E^b = A_E.$$

- (iii) Prove that if the wholesale water price goes up, then the Egyptian firm uses less water.

Answer. Pick any prices $(p^x, p_E^b; w_E, p^a)$, and let $(\bar{A}_E^x, \bar{L}_E^x, \bar{A}_E^b, \bar{L}_E^b)$ be an optimal choice there. By the envelope theorem,

$$\begin{aligned} & \frac{\partial \pi_E(p^x, p_E^b; w_E, p^a)}{\partial p^a} \\ &= \left[\frac{\partial}{\partial p^a} \{p^x f(A_E^x, L_E^x) + p_E^b g(A_E^b, L_E^b) - p^a(A_E^x + A_E^b) - w_E(L_E^x + L_E^b)\} \right]_{(A_E^x, L_E^x, A_E^b, L_E^b) = (\bar{A}_E^x, \bar{L}_E^x, \bar{A}_E^b, \bar{L}_E^b)} \\ &= [-(A_E^x + A_E^b)]_{(A_E^x, L_E^x, A_E^b, L_E^b) = (\bar{A}_E^x, \bar{L}_E^x, \bar{A}_E^b, \bar{L}_E^b)} \\ &= -\bar{A}_E, \end{aligned}$$

where $\bar{A}_E = \bar{A}_E^x + \bar{A}_E^b$ is the Egyptian firm's wholesale water demand. Since this relationship holds for all possible prices, we deduce that

$$\frac{\partial \pi_E(p^x, p_E^b; w_E, p^a)}{\partial p^a} = -A_E(p^x, p_E^b; w_E, p^a),$$

where A_E is the wholesale water factor demand function.

Notice that π_E is a convex function, since it is the upper envelope of all linear (and hence convex) functions, one for each possible choice. Thus, the left side of this equation is increasing in the price of wholesale water, p^a . It follows that the right side is also increasing, which means that A_E is decreasing in p^a . We conclude that the Egyptian firm's demand for wholesale water is decreasing in the price.

Part B

- (i) (easy) Consider a metric space (X, d) , and two sets U and Y with $U \subseteq Y \subseteq X$. Prove that if U is open in (X, d) , then U is open inside (Y, d) .

Answer. Pick any $x \in U$. We need to prove that there exists some radius $r > 0$ such that the open ball $B_r^Y(x) = \{x' \in Y : d(x', x) < r\}$ is a subset of U .

Since U is open inside (X, d) , it follows that there is some radius $r > 0$ such that $B_r^X(x) = \{x' \in X : d(x', x) < r\}$ is a subset of U . Since $B_r^X(x) \subseteq U$, it follows that $B_r^X(x) = B_r^Y(x)$. We conclude that $B_r^Y(x) \subseteq U$, as required.

- (ii) (easy) Two countries are bargaining over a truce agreement x which can be chosen from a compact metric space (X, d) . The countries' utility functions $u_1, u_2 : X \rightarrow [0, 1]$ are continuous. Let A be the set of agreements for which country 1 is strictly better off than country 2, i.e. $A = \{x \in X : u_1(x) > u_2(x)\}$. Prove A is open.

Answer. Let $v(x) = u_1(x) - u_2(x)$, which is continuous. Then $A = v^{-1}(\mathbb{R}_{++})$. Since A is the pre-image of an open set \mathbb{R}_{++} , it is open.

(iii) (easy) Find a counter-example to this false conjecture: $\text{int}(\text{cl}(\text{int}(A))) = \text{int}(A)$.

Answer. Let $(X, d) = ([0, 1], d_2)$ and $A = (0, 1)$. Then $\text{int}(A) = (0, 1)$ and $\text{int}(\text{cl}(\text{int}(A))) = [0, 1]$.

(iv) (easy) Suppose the state space of an infinite horizon dynamic programming problem is $X = \mathbb{R}_{++} \times \{0, 1\}$. Is the metric space of possible value functions, $(B(X), d_\infty)$, a complete metric space?

Answer. Yes. In class, we established that any function space $(B(X, Y), d_\infty)$ is complete if the co-domain (Y, d_Y) is complete. The co-domain in this case is (\mathbb{R}, d_2) , which is complete.

(v) (medium) Consider any two metric spaces (X, d) and (X, d') . Suppose that for any $x^0 \in X$, the function $f(x) = d'(x, x^0)$ is a continuous function from (X, d) to (\mathbb{R}_+, d_2) . Prove that if A is open in (X, d') , then A is open in (X, d) .

Answer. Step 1. If x_n converges to x^* according to d , then it also converges according to d' . Suppose $x_n \rightarrow x^*$ inside (X, d) . Let $g(x) = d'(x, x^*)$, which is assumed to be a continuous function from (X, d) to (\mathbb{R}_+, d_2) . Since g is continuous, $g(x_n) \rightarrow g(x^*)$. Now, $g(x_n) = d'(x_n, x^*)$ and $g(x^*) = 0$. Thus, $d'(x_n, x^*) \rightarrow 0$ and we conclude that $x_n \rightarrow x^*$ inside (X, d') .

Step 2. If A is closed according to d' , then it is also closed according to d . Pick any convergent sequence $a_n \in A$ converging to a^* according to d . We want to prove that $a^* \in A$.

Since $a_n \rightarrow a^*$ according to d , from step 1 we know that $a_n \rightarrow a^*$ according to d' . Since A is closed according to d' , it follows that $a^* \in A$.

Step 3. If A is open according to d' , then it is also open according to d . If A is open according to d' , then $B = X \setminus A$ is closed according to d' . By step 2, B is closed according to d , and thus $A = X \setminus B$ is open according to d .

(vi) (medium) Consider a function $f : X \rightarrow Y$ where (X, d_X) and (Y, d_Y) are metric spaces. Consider the metric space $Z = (X \times Y, d_Z)$, where $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$. Let $A \subseteq Z$ be the set $\{(x, f(x)) : x \in X\}$, which is called the *graph* of f . Prove that if f is continuous then A is closed.

Comment. This theorem is called the Closed Graph Theorem.

Answer. Pick any convergent sequence $(x_n, y_n) \in A$ with $(x_n, y_n) \rightarrow (x^*, y^*)$. It follows that $x_n \rightarrow x^*$ and $y_n \rightarrow y^*$. Since f is continuous, $y^* = f(x^*)$. So $(x^*, y^*) \in A$. Since the choice of convergent sequence was arbitrary, A is closed.

(vii) (medium) Suppose $f : X \rightarrow Y$, (X, d_X) , (Y, d_Y) , (Z, d_Z) and A are defined as in the previous question. Prove that if f is continuous and (X, d_X) is connected, then A is connected. *Hint: Consider the function $g(x) = (x, f(x))$.*

Answer. Consider the function $g : X \rightarrow Z$ defined by $g(x) = (x, f(x))$. Notice that g is continuous, since whenever $x_n \rightarrow x^*$, $(x_n, f(x_n)) \rightarrow (x^*, f(x^*))$. This is because $d_X(x_n, x^*) \rightarrow 0$ and $d_Y(f(x_n), f(x^*)) \rightarrow 0$ imply that $d_Z(x_n, f(x_n); x^*, f(x^*)) = d_X(x_n, x^*) + d_Y(f(x_n), f(x^*)) \rightarrow 0$.

Next, notice that $A = g(X)$ which means that A is the range of a continuous function with a connected domain. It follows that (A, d_Z) is connected.

- (viii) (hard) Consider the metric spaces (X, d) and (Y, d) where $Y \subseteq X$. Prove that if U is open inside (Y, d) , then there exists an open set V inside (X, d) such that $U = V \cap Y$.

Answer. Since U is open inside (Y, d) , every point $u \in U$ has an open ball inside (Y, d) of radius $r(u)$ such that $B_{r(u)}^Y(u) \subseteq U$. Let $V = \cup_{u \in U} B_{r(u)}^X(u)$ be the union of balls of these same centres and radii, but inside (X, d) .

Notice that V is open inside (X, d) , since it is the union of open sets.

Also notice that each ball can be constructed as $B_{r(u)}^Y(u) = B_{r(u)}^X(u) \cap Y$. It follows that

$$U = \cup_{u \in U} B_{r(u)}^Y(u) = \cup_{u \in U} [B_{r(u)}^X(u) \cap Y] = [\cup_{u \in U} B_{r(u)}^X(u)] \cap Y = V \cap Y,$$

as required.

49: Micro 1, December 2022

Suppose there are two sources of energy, gas and solar power electricity. For heating homes, the two sources are perfect substitutes. But for manufacturing, they have different uses and are imperfect substitutes. Each household is endowed with gas deposits and solar panels. Households sell solar power directly to each other and to factories, and gas that they sell on the wholesale gas market. Households also sell their labour inelastically. Households buy appliances, electricity and retail gas. Factories use labour, gas and solar power to make appliances. The gas firm uses labour and wholesale gas to make retail gas.

- (i) Formulate a competitive model of the wholesale gas, retail gas, solar power, labour and appliance markets.

Answer. Households. There are n identical households, endowed with 1 unit of time, 1 unit of wholesale gas and e units of solar power, which it sells at prices w , t^g , and r^s respectively. The household also receives dividends π/n . It then buys g units of gas at price r^g and s units of solar energy at price r^s to heat the home, and a appliances at price p . This gives the household a utility of $u(a, g + s)$. Its utility maximisation problem is

$$\begin{aligned} \max_{a, g, s} u(a, g + s) \\ \text{s.t. } pa + r^g g + r^s s = w + t^g + er^s + \pi/n. \end{aligned}$$

Gas firm. The gas firm buys H^g units of labour, G^g units of wholesale gas and produces $f^g(H^g, G^g)$ units of retail gas. Its profit function is

$$\pi^g(r^g; t^g, w) = \max_{H^g, G^g} r^g f^g(H^g, G^g) - wH^g - t^g G^g.$$

Appliance firm. The appliance firm buys H^a units of labour, G^a units of retail gas and S^a units of solar power and produces $f^a(G^a, S^a, H^a)$ appliances. Its profit function is

$$\pi^a(p; r^g, r^s, w) = \max_{G^a, S^a, H^a} pf^a(G^a, S^a, H^a) - wH^a - r^g G^a - r^s S^a.$$

Equilibrium. Prices (w, p, t^g, r^g, r^s) and quantities $(a, g, s, H^g, G^g, G^a, S^a, H^a)$ form an equilibrium if all quantities solve their respective optimisation problem, and all markets clear, i.e.

$$\begin{aligned} n &= H^g + H^a \\ na &= f^a(G^a, S^a, H^a) \\ n &= G^g \\ ng + G^a &= f^g(H^g, G^g) \\ ns + S^a &= ne \end{aligned}$$

- (ii) Prove that Walras law holds in the context of your model. (Recall: that Walras law says that the market value of the excess demand at market prices is zero, even if those prices do not lead to an equilibrium.) *Hint: substitute the profit functions into the budget constraint.*

Answer. Assume that u is strictly increasing, so that the households' budget constraint holds with equality. Fix prices (w, p, t^g, r^g, r^s) , which need not be equilibrium prices. Let (a, g, s) , (H^g, G^g) and (G^a, S^a, H^a) be the household's and firms' optimal choices. The households' budget constraint is:

$$(w, p, t^g, r^g, r^s) \cdot (0, a, 0, g, s) - (1, 0, 1, 0, e) - \pi/n = 0.$$

Simplifying and adding up across all households, this becomes

$$n(w, p, t^g, r^g, r^s) \cdot (-1, a, -1, g, s - e) - \pi = 0.$$

Moreover, firm profits are

$$\begin{aligned} \pi &= (w, p, t^g, r^g, r^s) \cdot [(-H^g, 0, -G^g, f^g(H^g, G^g), 0) + (-H^a, f^a(G^a, S^a, H^a), 0, -G^a, -S^a)] \\ &= (w, p, t^g, r^g, r^s) \cdot (-H^g - H^a, f^a(G^a, S^a, H^a), -G^g, f^g(H^g, G^g) - G^a, -S^a) \end{aligned}$$

Substituting the firm profits into the summed budget constraints gives:

$$(w, p, t^g, r^g, r^s) \cdot (H^g + H^a - n, na - f^a(G^a, S^a, H^a), G^g - n, ng - f^g(H^g, G^g), S^a + ns - ne) = 0.$$

The first vector consists of prices. The second vector consists of the excess demands in each market – compare them to the market clearing conditions above. We have established the market value of these excess demands equals 0, as required.

- (iii) Suppose the factories and the electricity firm merge into a single company. Prove that the merged company's demand for wholesale gas decreases if the price of wholesale gas increases.

Answer. Only the gas firm buys wholesale gas, so it suffices to show that the gas firm's wholesale gas demand is decreasing in wholesale gas prices. Let (\hat{H}^g, \hat{G}^g) be optimal choices for the gas firm. By the envelope theorem,

$$\begin{aligned} & \frac{\partial}{\partial t^g} \pi^g(r^g; t^g, w) \\ &= \left[\frac{\partial}{\partial t^g} \{r^g f^g(H^g, G^g) - wH^g - t^g G^g\} \right]_{(H^g, G^g) = (\hat{H}^g, \hat{G}^g)} \\ &= [-G^g]_{(H^g, G^g) = (\hat{H}^g, \hat{G}^g)} \\ &= -\hat{G}^g. \end{aligned}$$

Since the prices were chosen arbitrarily, we deduce that

$$\frac{\partial}{\partial t^g} \pi^g(r^g; t^g, w) = -G^g(r^g; t^g, w).$$

Now, the profit function $\pi^g(r^g; t^g, w)$ is the upper envelope of a set of linear (and hence convex) functions, one function for each possible choice of (H^g, G^g) . So π^g is

a convex function. It follows that the derivative of the profit function on the left side of the equation is increasing. We deduce that the right side is increasing, and conclude that the wholesale gas demand function $G^g(r^g; t^g, w)$ is decreasing in the wholesale gas price t^g .

- (iv) Assume that gas and electricity are normal goods for households, and the utility and production functions are strictly concave. Suppose that there are global warming protests, and that half of the population protest. The protesting households do not sell (or use) any gas. Prove the following:

- (a) There is at most one equilibrium without the protests. (The same logic applies when there are protests.)

Answer. By the first welfare theorem, every equilibrium is efficient. Since every household is identical, every equilibrium is symmetric. Thus, any efficient allocation solves

$$\begin{aligned} & \max_{a, g, s, H^g, G^g, G^a, S^a, H^a} u(a, g + s) \\ & \text{s.t.} \\ & n = H^g + H^a \\ & na = f^a(G^a, S^a, H^a) \\ & n = G^g \\ & ng + G^a = f^g(H^g, G^g) \\ & ns + S^a = ne. \end{aligned}$$

Since the feasible choice set is convex and the objective is strictly concave, this has a unique solution. We conclude that there is at most one equilibrium.

- (b) During protests, protestors heat their homes less than non-protestors.

Answer. Since protestors do not use or sell their gas, the t^g term on the budget constraint is zero for them. Thus, protestor income is lower than non-protestor income. Apart from this, their optimisation problems are the same. Since we assumed gas and electricity are normal goods, the protestor households demand less of both.

- (c) It is possible to devise a lump-sum tax scheme that makes the protestors heat their homes more, and the non-protestors heat their homes less. (Assume this satisfies the protestors, so they stop protesting.)

Answer. Suppose the protestors receive a lump-sum transfer of T and non-protestors pay a lump-sum tax of T . These transfers are budget balanced with $T \frac{n}{2} - T \frac{n}{2} = 0$.

Since the protestors became wealthier, and gas and electricity are normal goods, we deduce that protestors demand more energy for heating their homes. Similarly, the non-protestors become poorer, and demand less energy for heating their homes.

50: AME, May 2023

Part A

In the early 19th century, Australia traded mostly with England. Australia exported wool and imported hardware. Households in both countries are endowed with labour. In addition, households in Australia are endowed with wool, and in England are endowed with hardware. Households buy homes and clothes. Homes are made from hardware and labour. Clothes are made from wool and labour.

- (i) Formulate a competitive equilibrium model of this economy.

Answer. Households. There are two locations $\ell \in \{A, E\}$, with a population of n_ℓ each. Households in each country are endowed with 1 unit of labour which they supply inelastically for a wage of w_ℓ , wool x_ℓ supplied at price p^x , and hardware y_ℓ supplied at price p^y . We assume $x_E = 0$ and $y_A = 0$. The households receive an equal share of the local firms' profits, π_ℓ/n_ℓ . Households buy h_ℓ homes and c_ℓ clothes at prices p_ℓ^h and p_ℓ^c , which gives them a utility of $u(h_\ell, c_\ell)$. The household utility maximisation problem in location ℓ is

$$\begin{aligned} & \max_{h_\ell, c_\ell} u(h_\ell, c_\ell) \\ \text{s.t. } & p_\ell^h h_\ell + p_\ell^c c_\ell = w_\ell + p^x x_\ell + p^y y_\ell + \pi_\ell/n_\ell. \end{aligned}$$

Firms. There is one firm in each location. Each firm chooses how much labour to allocate to clothes N_ℓ^c and to homes N_ℓ^h , how much wool to use X_ℓ and how much hardware to use Y_ℓ . The firm produces $f(N_\ell^c, X_\ell)$ clothes and $g(N_\ell^h, Y_\ell)$ homes. The firm's profit function is

$$\begin{aligned} & \pi_\ell(p_\ell^h, p_\ell^c; w_\ell, p^x, p^y) \\ & = \max_{N_\ell^c, N_\ell^h, X_\ell, Y_\ell} p_\ell^h f(N_\ell^h, Y_\ell) + p_\ell^c g(N_\ell^c, X_\ell) - w_\ell(N_\ell^h + N_\ell^c) - p^x X_\ell - p^y Y_\ell. \end{aligned}$$

Equilibrium. Prices $(p_A^h, p_A^c, w_A, p_E^h, p_E^c, w_E, p^x, p^y)$ and quantities $(h_\ell, c_\ell, N_\ell^c, N_\ell^h, X_\ell)$ constitute an equilibrium if the quantities solve the respective problems above, and all markets clear, i.e.

$$\begin{aligned} n_A x_A &= X_A + X_E \\ n_E y_E &= Y_A + Y_E \\ n_\ell h_\ell &= f(N_\ell^h, Y_\ell) \text{ for } \ell \in \{A, E\} \\ n_\ell c_\ell &= g(N_\ell^c, X_\ell) \text{ for } \ell \in \{A, E\} \\ n_\ell &= N_\ell^c + N_\ell^h \text{ for } \ell \in \{A, E\}. \end{aligned}$$

- (ii) Suppose all production in Australia is managed by the East India Company's Australian division. Formulate the division's profit maximisation problem in which it chooses its aggregate labour demand first, and the other choices are buried inside a value function.

Answer. We can rewrite the Australian firm's profit function as

$$\pi_A(p_A^h, p_A^c; w_A, p^x, p^y) = \max_{N_A} V_A(p_A^h, p_A^c; p^x, p^y, N_A) - w_A N_A$$

where

$$\begin{aligned} V_A(p_A^h, p_A^c; p^x, p^y, N_A) &= \max_{N_A^c, N_A^h, X_A, Y_A} p_A^h f(N_A^h, Y_A) + p_A^c g(N_A^c, X_A) - p^x X_A - p^y Y_A \\ \text{s.t. } N_A^h + N_A^c &= N_A. \end{aligned}$$

(iii) Prove that when English wages increase, the demand for English labour decreases.

Answer. Fix local market prices $(p_E^h, p_E^c; w_E, p^x, p^y)$. Let $(\bar{N}^c, \bar{N}^h, \bar{X}, \bar{Y})$ be optimal choices at these prices. By the envelope theorem,

$$\begin{aligned} &\frac{\partial}{\partial w_E} \pi_E(p_E^h, p_E^c; w_E, p^x, p^y) \\ &= \left[\frac{\partial}{\partial w_E} (p_E^h f(N_E^h, Y_E) + p_E^c g(N_E^c, X_E) - w_E(N_E^h + N_E^c) - p^x X_E - p^y Y_E) \right]_{(\bar{N}^c, \bar{N}^h, \bar{X}, \bar{Y})} \\ &= [-(N_E^h + N_E^c)]_{(\bar{N}^c, \bar{N}^h, \bar{X}, \bar{Y})} \\ &= -\bar{N}^h - \bar{N}^c. \end{aligned}$$

Let $N(p_E^h, p_E^c; w_E, p^x, p^y)$ be the English labour demand. We have established that

$$\frac{\partial}{\partial w_E} \pi_E(p_E^h, p_E^c; w_E, p^x, p^y) = -N(p_E^h, p_E^c; w_E, p^x, p^y).$$

Notice that π_E is the upper envelope of linear functions in prices, with one function for each combination of choices of (N_E^c, N_E^h, X_E, Y_E) . Since linear functions are convex, π_E is the upper envelope of convex functions, so π_E is a convex function. Thus $\frac{\partial \pi_E}{\partial w_E}$ is increasing in w_E .

Since the left side of the envelope formula is increasing, so is the right side. We deduce that English labour demand $N(p_E^h, p_E^c; w_E, p^x, p^y)$ is decreasing in English wages w_E .

Part B

(i) (easy) Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are Lipschitz continuous of degree $a < 1$. Prove that $h(x) = g(f(x))$ is a contraction of degree a^2 .

Answer. Let d_X and d_Y be the metrics for the two spaces. The condition that f and g are Lipschitz continuous of degree a means

$$\begin{aligned} d_Y(f(x), f(x')) &\leq a d_X(x, x') \\ d_X(g(y), g(y')) &\leq a d_Y(y, y'). \end{aligned}$$

Combining, we deduce

$$d_X(h(x), h(x')) = d_X(g(f(x)), g(f(x'))) \leq a d_Y(f(x), f(x')) \leq a^2 d_X(x, x').$$

Thus, $h : X \rightarrow X$ is a contraction of degree a^2 .

- (ii) (easy) Suppose U and V are open sets inside (X, d_X) and (Y, d_Y) respectively. Prove that $U \times V$ is open inside $(X \times Y, d_Z)$ where $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$.

Answer. Pick any point $z^* = (x^*, y^*) \in U \times V$. Since U is open, there exists a radius $r > 0$ such that $B_r(x^*) \subseteq U$. Similarly, there is a radius $s > 0$ such that $B_s(y^*) \subseteq V$. Let $t = \min\{r, s\}$.

To establish that $U \times V$ is open, it suffices to show that $B_t(z^*) \subseteq U \times V$. Pick any $z' = (x', y') \in B_t(z^*)$. Then $d_Z(z, z^*) = d_X(x', x^*) + d_Y(y', y^*)$. Since $d_Z(z, z^*) < t$ it follows that $d_X(x', x^*) < r$ and $d_Y(y', y^*) < s$. In other words, $x' \in B_r(x^*) \subseteq U$ and $y' \in B_s(y^*) \subseteq V$. We deduce that $z' \in U \times V$. Since z' was chosen arbitrarily from $B_t(z^*)$, we conclude that $B_t(z^*) \subseteq U \times V$.

- (iii) (easy) Let A and B be sets inside (X, d_X) and (Y, d_Y) respectively. Prove that $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ inside $(X \times Y, d_Z)$, where $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$.

Answer. First, we prove that $\text{cl}(A \times B) \subseteq \text{cl}(A) \times \text{cl}(B)$. Suppose $(a^*, b^*) \in \text{cl}(A \times B)$. This means there is a sequence $(a_n, b_n) \in A \times B$ with $(a_n, b_n) \rightarrow (a^*, b^*)$. Hence $d_Z(a_n, b_n; a^*, b^*) \rightarrow 0$. It follows that $d_X(a_n, a^*) \rightarrow 0$ and $d_Y(b_n, b^*) \rightarrow 0$, and hence $a_n \rightarrow a^*$ and $b_n \rightarrow b^*$. Thus $a^* \in \text{cl}(A)$ and $b^* \in \text{cl}(B)$. We conclude that $(a^*, b^*) \in \text{cl}(A) \times \text{cl}(B)$.

Second, we prove that $\text{cl}(A) \times \text{cl}(B) \subseteq \text{cl}(A \times B)$. Pick any $a^* \in \text{cl}(A)$, which means there is some sequence $a_n \in A$ with $a_n \rightarrow a^*$. Hence $d_X(a_n, a^*) \rightarrow 0$. Similarly, pick any $b^* \in \text{cl}(B)$, which means there is some $b_n \in B$ with $d_Y(b_n, b^*) \rightarrow 0$. It follows that $d_Z(a_n, b_n; a^*, b^*) \rightarrow 0$. This implies that $(a_n, b_n) \rightarrow (a^*, b^*)$. We conclude that $(a^*, b^*) \in \text{cl}(A \times B)$.

- (iv) (medium) Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \rightarrow X$ is a continuous self-map that has no fixed points. Suppose that there is a bijection g such that $g : X \rightarrow Y$ and its inverse $g^{-1} : Y \rightarrow X$ are continuous. Prove that (Y, d_Y) has a continuous self-map that has no fixed points.

Answer. Let $a : Y \rightarrow Y$ be the function $a(y) = g(f(g^{-1}(y)))$. Suppose for the sake of contradiction that y^* is a fixed point of a . Let $x^* = g^{-1}(y^*)$. Then $f(x^*) = f(g^{-1}(y^*)) = g^{-1}(g(f(g^{-1}(y^*)))) = g^{-1}(a(y^*)) = g^{-1}(y^*) = x^*$, so x^* is a fixed point of f . But this contradicts the condition that f has no fixed points.

- (v) Consider the following version of Pavoni's (2009) model of unemployment insurance. His notation is as follows (it is unnecessary to answer the questions). U is lifetime utility promised to the unemployed person, U^u is the future promise if the worker remains unemployed tomorrow, U^e is the future promise if the worker finds a job by tomorrow, $W(U^e)$ is the government's value of fulfilling this second promise, π is the probability of finding a job, b is the unemployment payment today, $u(b)$ is the person's utility of receiving a payment of b where $u \in B(\mathbb{R}_+)$, β is the discount rate, and $V(U)$ is the government's value of promising $U \in X \subset \mathbb{R}$ where $V \in B(X)$,

which is the solution to the Bellman equation

$$\begin{aligned} V(U) &= \sup_{b, U^e, U^u} -b + \beta[\pi W(U^e) + (1 - \pi)V(U^u)] \\ \text{s.t. } U &= u(b) + \beta[\pi U^e + (1 - \pi)U^u] \\ &\text{and } U^e \geq U^u. \end{aligned}$$

The first constraint says that the government's promise U can be fulfilled by a combination of paying the person today, or making more promises. The second constraint says that the person would prefer to accept all job offers.

- (a) (medium) Suppose that the Bellman operator T is a contraction on $(B(X), d_\infty)$. Prove that if u is increasing, then V is (weakly) decreasing.

Answer. We will make use of the following claim:

Claim. If $g(x) = \sup_y f(x, y)$ and f is decreasing in x , then g is decreasing in x .

Proof. We just prove the case in which the suprema are attained as maxima. (Generalising is fairly straight forward, but hard to read.)

Suppose $x < x'$, and pick y and y' so that $g(x) = f(x, y)$ and $g(x') = f(x', y')$. We have

$$g(x) = f(x, y) \geq f(x, y') \geq f(x', y') = g(x'),$$

as required. □

We can write the Bellman operator $T : B(X) \rightarrow B(X)$ as

$$\begin{aligned} T(V')(U) &= \sup_{U^e, U^u} -u^{-1}(U - \beta[\pi U^e + (1 - \pi)U^u]) + \beta[\pi W(U^e) + (1 - \pi)V'(U^u)] \\ \text{s.t. } U^e &\geq U^u. \end{aligned}$$

Recall that $(B(X), d_\infty)$ is a complete metric space. By Banach's fixed point theorem, T has a unique fixed point, which we call V .

Notice that U appears only once in the objective. Since u is increasing, u^{-1} is also increasing, and thus $-u^{-1}$ is decreasing. We deduce that for each (U^e, U^u) , the objective is decreasing in U .

Since the objective is decreasing in U , the claim implies that the value function $T(V)$ is decreasing in U . (Note that the constraint does not depend on U – it merely specifies the choice set.)

Let $D = \{f \in B(X) : f \text{ is decreasing}\}$. We have just established that T is a self-map on D .

Recall that (D, d_∞) is a complete metric space. Since T is a contraction, Banach's fixed point theorem establishes that there is a unique fixed point $V^* \in D$. Since V is the only solution to the Bellman equation (see above), we deduce that $V = V^*$. We conclude that V is decreasing.

- (b) (medium) Suppose that the Bellman operator T is a contraction on $(CB(X), d_\infty)$ and that the range of u is compact. Prove that there exists an optimal choice of (b, U^e, U^u) for all promises $U \in X$.

Answer. Like before, we can write the Bellman operator $T : CB(X) \rightarrow CB(X)$ as

$$T(V')(U) = \sup_{(b, U^e, U^u) \in M(U)} -b + \beta[\pi W(U^e) + (1 - \pi)V'(U^u)],$$

where

$$M(U) = \{(b, U^e, U^u) \in \mathbb{R}_+ \times X^2 : U^e \geq U^u, U = u(b) + \beta[\pi U^e + (1 - \pi)U^u]\}.$$

Recall that $(CB(X), d_\infty)$ is a complete metric space. By Banach's fixed point theorem, T has a unique fixed point, which we call V .

Note that V is continuous, since $V \in CB(X)$.

Since $V : X \rightarrow \mathbb{R}$, the question (implicitly) assumes that $M(U)$ is non-empty for all $U \in X$.

Moreover, $M(U)$ is compact. To see this, let

$$N = \{(U^e, U^u) \in X \times X : U^e \geq U^u\}.$$

Notice that N is a compact set, since N is a closed subset of $X \times X$, which is compact. Since $M(U) = f(N)$ is the image of a continuous function

$$f(U^e, U^u) = (u^{-1}(U - \beta[\pi U^e + (1 - \pi)U^u]), U^e, U^u),$$

we conclude that $M(U)$ is compact.

Since the objective is continuous and the choice set is non-empty and compact, the extreme value theorem establishes that there is an optimal choice.

- (vi) (hard) Let (X, d) be a compact metric space. Suppose $f : X \times [0, 1] \rightarrow \mathbb{R}$ is continuous, with distances in the domain measured by

$$d'(x, y; x', y') = d(x, x') + d_2(y, y').$$

Let $g_n : X \rightarrow X$ be defined by $g_n(x) = f(x, 1/n)$. Prove that g_n is a convergent sequence inside the metric space $(CB(X), d_\infty)$.

Comment. This question is related to *homotopies*, which are beyond the scope of the course. But if you are curious: f is an example of *homotopy*, and g_1 and g^* (defined below) are *homotopic*. This means that g_1 can be continuously deformed to make g^* , and vice versa.

Homotopies are sometimes used to calculate equilibria of complicated economies by starting with a simple economy, and gradually making it more complicated. For example, each step might use the solution from the previous step as an initial guess. See Herings and Peeters (2009) and Kubler and Schmedders (2000).

Answer. Let $g^*(x) = f(x, 0)$. We will prove that $g_n \rightarrow g^*$.

Since f is continuous, it follows that g^* and g_n are continuous.

Notice that $d_\infty(g_n, g^*) = \sup_{x \in X} d_2(g_n(x), g^*(x))$. Since the objective is continuous and the domain is compact, the extreme value theorem implies that this

optimisation problem has a solution, x_n . Thus we deduce that $d_\infty(g_n, g^*) = d_2(g_n(x_n), g^*(x_n))$.

Since (X, d) is compact, x_n has a convergent subsequence $x_{k_n} \rightarrow x^*$. Thus, $d_\infty(g_{k_n}, g^*) = d_2(g_{k_n}(x_{k_n}), g^*(x_{k_n}))$.

Since g^* is continuous, we deduce that $g^*(x_{k_n}) \rightarrow g^*(x^*)$. Similarly, $g_{k_n}(x_{k_n}) = f(x_{k_n}, 1/k_n) \rightarrow f(x^*, 0) = g^*(x^*)$.

We conclude that $d_\infty(g_{k_n}, g^*) \rightarrow 0$, as required.

51: Micro 1, May 2023

In the “War of the currents” in the late 1800s, George Westinghouse’s and Thomas Edison’s companies provided competing electricity distribution systems. Westinghouse’s alternating current (AC) can power motors such as refrigerators, whereas Edison’s direct current (DC) can power semiconductors such as computers. Rectifiers and inverters can convert between AC and DC. AC won the war, so laptops come with power adapters with rectifiers, but refrigerators do not need inverters.

Suppose Westinghouse owns an AC supplier and Edison owns a DC supplier. (The other firms are held by the rest of the population.) They produce electricity from labour. The first unit needs a lot of labour, so it is inefficient for both firms to operate.

The four electrical goods – computers, refrigerators, rectifiers and inverters – are produced from labour. Households supply labour inelastically, and buy electrical goods and appropriate electricity to power them. Households derive utility from using computers and refrigerators, which are always turned on.

- (i) Formulate a competitive model of the labour, electricity and electrical goods markets (7 in all).

Answer.

Households. The set of households H includes workers H_W and entrepreneurs $H_E = \{JW, TE\}$. Each household $h \in H$ sells its labour endowment ℓ_h at wage w , and chooses how many fridges q_h^f , computers q_h^c , inverters q_h^i , rectifiers q_h^r , and how much DC power q_h^{DC} and AC power q_h^{AC} to buy at prices p^f , p^c , p^i , p^r , p^{DC} , p^{AC} , respectively. Working computers and fridges give the household a utility of $u(q_h^f, q_h^c)$. Household h receives dividends d_h , where

$$d_h = \begin{cases} \pi^{AC} & \text{if } h = JW, \\ \pi^{DC} & \text{if } h = TE, \\ \frac{1}{|H_W|}(\pi^f + \pi^c + \pi^i + \pi^r) & \text{if } h \in H_W. \end{cases}$$

The household’s utility maximisation problem is

$$\begin{aligned} & \max_{q_h^f, q_h^c, q_h^i, q_h^r, q_h^{DC}, q_h^{AC}} u(q_h^f, q_h^c) \\ \text{s.t. } & p^f q_h^f + p^c q_h^c + p^i q_h^i + p^r q_h^r + p^{DC} q_h^{DC} + p^{AC} q_h^{AC} = w \ell_h + d_h \\ & q_h^c = q_h^{DC} + q_h^r - q_h^i \\ & q_h^f = q_h^{AC} + q_h^i - q_h^r. \end{aligned}$$

Firms. Firm $x \in X = \{f, c, i, r, AC, DC\}$ uses L^x workers to make $g^x(L^x)$ units of item x . The profit function is

$$\pi^x(p^x; w) = \max_{L^x} p^x g^x(L^x) - w L^x.$$

Equilibrium. Prices $(w, p^f, p^c, p^i, p^r, p^{AC}, p^{DC})$ and quantities $(\ell_h, q_h^x, L_h^x)_{(x,h) \in X \times H}$ constitute an equilibrium of the quantities solve the respective optimisation problem

above, and each market clear, i.e.

$$\begin{aligned}
\sum_{h \in H} \ell_h &= \sum_{x \in X} L^x \\
\sum_{h \in H} q_h^f &= g^f(L^f) \\
\sum_{h \in H} q_h^c &= g^c(L^c) \\
\sum_{h \in H} q_h^r &= g^r(L^r) \\
\sum_{h \in H} q_h^i &= g^i(L^i) \\
\sum_{h \in H} q_h^{\text{AC}} &= g^{\text{AC}}(L^{\text{AC}}) \\
\sum_{h \in H} q_h^{\text{DC}} &= g^{\text{DC}}(L^{\text{DC}}).
\end{aligned}$$

- (ii) Prove that either AC or DC wins, i.e. in every equilibrium, either AC or DC are inactive.

Answer. The question states that it is inefficient to operate both AC and DC power (due to high fixed costs). By the first welfare theorem, every equilibrium is efficient. Thus, in every equilibrium, either the AC or DC markets are inactive.

- (iii) Edison is worried that Westinghouse is going to win, so he proposes a merger of their firms. Write the profit function of the merged firm.

Answer. The profit function of the merged firm can be written as

$$\pi^m(p^{\text{AC}}, p^{\text{DC}}; w) = \pi^{\text{AC}}(p^{\text{AC}}; w) + \pi^{\text{DC}}(p^{\text{DC}}; w).$$

- (iv) Edison has a new idea. He could offer a DC package deal that gives the same utility as before. Prove that if the price of inverters increases, then Edison's package includes fewer inverters.

Answer. Let \bar{u} be the status quo utility. The cost of the cheapest DC package deal is given by the following expenditure function,

$$\begin{aligned}
e(p^f, p^c, p^i, p^{\text{DC}}) &= \min_{q^f, q^c, q^i, q^{\text{DC}}} p^f q^f + p^c q^c + p^i q^i + p^{\text{DC}} q^{\text{DC}} \\
\text{s.t. } u(q^f, q^c) &\geq \bar{u} \\
q^c &= q^{\text{DC}} - q^i \\
q^f &= q^i.
\end{aligned}$$

Let $(\bar{q}^f, \bar{q}^c, \bar{q}^i, \bar{q}^{\text{DC}})$ be Edison's proposed package deal given prices $(p^f, p^c, p^i, p^{\text{DC}})$.

By the envelope theorem,

$$\begin{aligned}
& \frac{\partial}{\partial p^i} e(p^f, p^c, p^i, p^{\text{DC}}) \\
&= \left[\frac{\partial}{\partial p^i} (p^f q^f + p^c q^c + p^i q^i + p^{\text{DC}} q^{\text{DC}}) \right]_{(q^f, q^c, q^i, q^{\text{DC}}) = (\bar{q}^f, \bar{q}^c, \bar{q}^i, \bar{q}^{\text{DC}})} \\
&= [q^i]_{(q^f, q^c, q^i, q^{\text{DC}}) = (\bar{q}^f, \bar{q}^c, \bar{q}^i, \bar{q}^{\text{DC}})} \\
&= \bar{q}^i.
\end{aligned}$$

In fact, since the prices were chosen arbitrarily, we can write

$$\frac{\partial}{\partial p^i} e(p^f, p^c, p^i, p^{\text{DC}}) = \bar{q}^i(p^f, p^c, p^i, p^{\text{DC}}).$$

Notice that e is the lower envelope of functions that is linear in prices, with one for each possible package deal. Since the lower envelope of linear (and hence concave) functions is concave, we deduce that e is a concave function.

Thus, the left side of the envelope formula is decreasing in p^i . The right side is also decreasing, so we conclude that the number of inverters included in the package deal is decreasing in the price of inverters.

- (v) Edison has another idea, lump-sum taxes. Suppose there are both AC and DC equilibria. What lump-sum taxes can the government use to implement the DC equilibrium? Would these improve Edison's DC firm's profits?

Answer. This is a trick question. Since there is a DC equilibrium $(p_{\text{DC}}, q_{\text{DC}})$ (written in short-hand), the lump-sum transfers are zero. Obviously, this policy has no impact. Conceptually, the second welfare theorem does not help coordinate among multiple equilibria. Thus, the policy does not improve Edison's profits.

52: Micro 1, December 2023

In 1918, the Hotel Saint Gellért in Budapest opened its doors. It was the first hotel to have a swimming pool. The architects (Hegedus, Sebestyen, and Sterk) had to choose the size of the swimming pool, the hotel room size, and the number of hotel rooms. Assume that all hotel rooms are identical. Each household supplies labour to build the hotel, and chooses how long to visit the hotel for. Each household owns an equal share in the hotel.

- (i) Formulate a competitive model of the hotel room and labour markets.

Hint 1: there is an infinite number of hotel room markets, one for each type of room (i.e. each combination of room and pool size). But only one is active.

Hint 2: you may assume that households can only demand one type of hotel room (of their choice).

Comment. The difficult part of this question was capturing the idea that there are many hotel room markets, as described in the first hint. All but one of these markets has both supply and demand equalling zero. Yet each of these markets has a price, which is important for determining which hotel market the developers will choose to enter. Few students got this right.

Answer. Hotel. There is one market for each hotel room size z and each swimming pool size s . Let p_{sz} denote the price of a hotel room in such a hotel, and p denotes the (infinite) matrix of prices of all types of hotel room. The hotel chooses the room size Z , the pool size S , and the number of workers L at wage w . It builds $f_{SZ}(L)$ hotel rooms of type SZ . (There is one production function for each (S, Z) combination.) The hotel's profit maximisation problem is

$$\pi(p; w) = \max_{S, Z, L} p_{SZ} f_{SZ}(L) - wL.$$

Households. There are n identical households. A representative household chooses how much labour ℓ to supply, which type (s, z) of hotel room to stay in, and for how long x . This gives the household a utility of $u(x, s, z, \ell)$. The utility maximisation problem is

$$\begin{aligned} & \max_{x, s, z, \ell} u(x, s, z, \ell) \\ & \text{s.t. } p_{sz}x = w\ell + \frac{\pi(p; w)}{n}. \end{aligned}$$

Equilibrium. Market prices (p, w) and quantities (S, Z, L, s, z, x, ℓ) form an equilibrium if the quantities solve the optimization problems above, and the markets clear, i.e.

$$\begin{aligned} I(s = \hat{s}, z = \hat{z})nx &= I(S = \hat{S}, Z = \hat{Z})f_{SZ}(L) \text{ for every } (\hat{s}, \hat{z}) \\ n\ell &= L. \end{aligned}$$

- (ii) Is it possible to normalise prices by dividing by the price of a type of hotel room that is not traded?

Answer. Yes. Pick any room type (s, z) that is not traded. As long as $p_{sz} > 0$, we can divide all other prices by p_{sz} and still have an equilibrium with the same allocation of resources. And notice that $p_{sz} > 0$ – otherwise there would be (infinite) demand for these rooms, contradicting the premise that (s, z) is not traded.

- (iii) The Société de l'Art Nouvea decided that the workers aren't working hard enough. Is it possible to design lump sum taxes to increase the aggregate hours worked?

Comment. A common mistake was to only consider lump sum transfers that treat all households equally.

Answer. Let (s, z, x, ℓ) be the equilibrium household quantities. If the production and utility functions are strictly increasing, then it is efficient to allocate all but one household to build the hotel, and the remaining household does not work, but is the only household to visit the hotel. Specifically, set $(\hat{s}, \hat{z}, \hat{x}, \hat{\ell}) = (s, z, 0, \ell(n + 1)/(n - 1))$ for households 1 to $n - 1$, and household n chooses $(s^*, z^*, x^*, \ell^*) = (s, z, f_{sz}((n - 1)\ell(n + 1)/(n - 1)), 0)$. This allocation involves more work, since $(n - 1)\ell(n + 1)/(n - 1) = \ell(n + 1) > \ell n$. By the second welfare theorem, this allocation of resources can be implemented.

- (iv) Prove that if the hotel room price increases (i.e. the price of the type of hotel room that the hotel supplies in equilibrium), then the hotel builds extra rooms.

Answer. Suppose (S^*, Z^*, L^*) are optimal choices given prices (p, w) . By the envelope theorem,

$$\begin{aligned} \frac{\partial}{\partial p_{SZ}} \pi(p; w) &= \left[\frac{\partial}{\partial p_{SZ}} \{p_{SZ} f_{SZ}(L) - wL\} \right]_{(S, Z, L) = (S^*, Z^*, L^*)} \\ &= [f_{SZ}(L)]_{(S, Z, L) = (S^*, Z^*, L^*)} \\ &= f_{S^* Z^*}(L^*). \end{aligned}$$

Since this is true for all (p, w) , this means that

$$\frac{\partial}{\partial p_{SZ}} \pi(p; w) = X(p; w),$$

where $X(p; w)$ is the firm's supply curve.

Notice that the profit function

$$\pi(p; w) = \max_{S, Z, L} p_{SZ} f_{SZ}(L) - wL$$

is the upper envelope of linear functions in prices, with one function for each (S, Z, L) . Since linear functions are convex, π is the upper envelope of convex functions. Therefore, π is a convex function.

Since π is convex, the left side of the envelope formula is increasing in $p_{S^* Z^*}$. Thus, the right side, $X(p; w)$ is increasing in $p_{S^* Z^*}$.

- (v) Suppose that at market prices (possibly non-equilibrium prices), all hotel room markets clear. Prove that the labour market clears.

Note: you will get more points if you show the details about how to adapt the logic from lectures.

Answer. Summing up the households' budget constraints gives

$$np_{sz}x = nw\ell + \pi(p; w).$$

Substituting in the profit function gives

$$np_{sz}x = nw\ell + p_{SZ}f_{SZ}(L) - wL.$$

The hotel room markets can only clear if the hotel and the households choose the same hotel sizes (i.e. $(s, z) = (S, Z)$) and if the supply equals demand ($nx = f_{SZ}(L)$). We deduce

$$np_{sz}x = nw\ell + p_{sz}nx - wL$$

which implies

$$0 = nw\ell - wL.$$

It follows that $n\ell = L$, as required.

53: AME, December 2023

Part A

Suppose an engineering firm designs and builds apartment buildings. It hires both full-time and part-time workers. Assume that full-time workers are more productive, because they complete urgent tasks more quickly, and are easier to reach to resolve problems. Assume that all households have two workers, and some households have children. Households with children have a stronger preference for part-time work. Households own the engineering firm, supply labour, and buy homes.

- (i) Formulate a competitive model of the three markets (the market for apartments, and full-time and part-time labour markets).

Comments. Common mistakes include:

- Assuming all households are identical.
- Assuming some households are “full-time” or “part-time” households, rather than families (with children) or not.
- Assuming that full-time and part-time workers do not collaborate, and hence have separate production functions.

Answer. Households. The set of households H is divided between families H_f and couples H_c . Household h supplies $\ell_{ft,h}$ and $\ell_{pt,h}$ units of full-time and part-time labour at wages w_{ft} and w_{pt} . Each household can supply up to two units of labour, so that $\ell_{ft,h} + \ell_{pt,h} \leq 2$. Each household also buys a home of size a at a price of p per square metre. This gives the household a utility of $u_h(a_h, \ell_{ft,h}, \ell_{pt,h})$, where families $h \in H_f$ value leisure time more. The household’s utility maximisation problem is

$$\begin{aligned} \max_{a_h, \ell_{ft,h}, \ell_{pt,h}} \quad & u_h(a_h, \ell_{ft,h}, \ell_{pt,h}) \\ \text{s.t.} \quad & pa_h \leq w_{ft}\ell_{ft,h} + w_{pt}\ell_{pt,h} + \pi/|H|, \\ & \text{and } \ell_{ft,h} + \ell_{pt,h} \leq 2. \end{aligned}$$

(Alternatively, the time constraint could be put inside the utility function.)

Engineering firm. The engineering firm hires workers L_{ft} full-time and L_{pt} part-time workers and makes $f(L_{ft}, L_{pt})$ square metres of apartments. The firm’s profit function is

$$\pi(p; w_{pt}, w_{ft}) = \max_{L_{ft}, L_{pt}} pf(L_{ft}, L_{pt}) - w_{ft}L_{ft,h} - w_{pt}L_{pt,h}$$

Equilibrium. Prices (p, w_{ft}, w_{pt}) and quantities $(a_h, \ell_{ft,h}, \ell_{pt,h}, L_{ft}, L_{pt})$ constitute an equilibrium if the quantities are optimal choices in the problems above, and all markets clear:

$$\begin{aligned} \sum_{h \in H} \ell_{ft,h} &= L_{ft} \\ \sum_{h \in H} \ell_{pt,h} &= L_{pt} \\ \sum_{h \in H} a_h &= f(L_{ft}, L_{pt}). \end{aligned}$$

- (ii) Prove that if the wages of part-time workers increases, then firm demands fewer part-time workers.

Answer. Suppose (L_{ft}^*, L_{pt}^*) maximises the firms profits when prices are (p, w_{pt}, w_{ft}) . By the envelope theorem,

$$\begin{aligned} & \frac{\partial}{\partial w_{pt}} \pi(p; w_{pt}, w_{ft}) \Big|_{(L_{ft}, L_{pt})=(L_{ft}^*, L_{pt}^*)} \\ &= \left[\frac{\partial}{\partial w_{pt}} \{pf(L_{ft}, L_{pt}) - w_{ft}L_{ft,h} - w_{pt}L_{pt,h}\} \right]_{(L_{ft}, L_{pt})=(L_{ft}^*, L_{pt}^*)} \\ &= [-L_{pt,h}]_{(L_{ft}, L_{pt})=(L_{ft}^*, L_{pt}^*)} \\ &= -L_{pt,h}^*. \end{aligned}$$

Since the profit function π is the upper envelope of functions that are linear in w_{pt} , it follows that π is convex in w_{pt} . We deduce that the left side of the envelope theorem equation is increasing in w_{pt} , and hence the right side is also increasing. We conclude that the part-time factor demand is decreasing in the part-time wage w_{pt} .

- (iii) Reformulate the firm's problem with a Bellman equation in which the only choice is the amount of apartment construction.

Answer.

$$\pi(p; w_{pt}, w_{ft}) = \max_y py - c(y; w_{pt}, w_{ft})$$

where

$$\begin{aligned} c(y; w_{pt}, w_{ft}) &= \min_{L_{ft}, L_{pt}} w_{ft}L_{ft,h} + w_{pt}L_{pt,h} \\ &\text{s.t. } f(L_{ft}, L_{pt}) \geq y. \end{aligned}$$

Part B

General remarks. Many students attempted to reformulate the problems via the contrapositive, or by doing a proof by contradiction. This is a good idea, but many students did not succeed in negating the statements correctly. For example, the contrapositive of "If (X, d) is unbounded then there exists an unbounded continuous function $f : X \rightarrow \mathbb{R}$ " is "If every continuous function $f : X \rightarrow \mathbb{R}$ is bounded, then (X, d) is bounded."

A common weakness was to only prove a special case of the question, e.g. to assume $(X, d) = (\mathbb{R}, d_2)$, when the question calls for full generality. For example, defining a function $f(x) = x^2$ only makes sense in a one-dimensional space like (\mathbb{R}, d_2) .

- (i) (easy) Provide a counter-example to the following false claim: If (X, d) is a metric space, and the interior of $A \subset X$ is connected, then A is connected.

Comment. Many students confused *interior* with *closure*.

Answer. Consider $(X, d) = (\mathbb{R}, d_2)$ and $A = [0, 1] \cup \{2\}$. We need to check two criteria:

- A is disconnected. This is true because $[0, 1]$ is both open and closed inside (A, d_2) .
- The interior of A is $(0, 1)$, which is connected. We proved that intervals are connected in class.

(ii) (easy) Consider the metric spaces (X, d_X) , (Y, d_Y) and (Z, d_Z) where $Z = X \times Y$ and $d_Z(x, y; x', y') = d_X(x, x') + d_Y(y, y')$. Prove that if (Z, d_Z) is connected, then (X, d_X) is connected.

Answer. Let $f : Z \rightarrow X$ be defined by $f(x, y) = x$. Notice that f is continuous because if $(x_n, y_n) \rightarrow (x^*, y^*)$, then $f(x_n, y_n) \rightarrow x^*$. In class, we proved that if the domain of a surjective and continuous function is connected, then the co-domain is connected. We conclude that (X, d_X) is connected.

(iii) (easy) Pick any set A inside a metric space (X, d) . Pick any radius $r > 0$ and let $U = \{x : (x, a) \in X \times A, d(x, a) < r\}$ be the set of all points in X that have a distance of less than r to some point inside A . Prove that U is an open set.

Answer. Notice that $U = \cup \{B_r(a) : a \in A\}$ is the union of all open balls of radius r centred at points inside A . Recall that unions of open sets are open, so U is an open set.

(iv) (easy) Suppose there are two bidders in an auction for the remnants of a bankrupt car factory. The first bidder values the factory at £20m. The first bidder spied on the second bidder, and knows he will bid £10m. Thus, his (expected) profit when bidding b million is

$$\pi(b) = \begin{cases} 0 & \text{if } b < 10, \\ 5 & \text{if } b = 10, \\ 20 - b & \text{if } b > 10. \end{cases}$$

Calculate the range $\pi(\mathbb{R})$, the maximum $\max \pi(\mathbb{R})$ and the supremum $\sup \pi(\mathbb{R})$, or prove that they do not exist.

Comment. Many students mistakenly assumed that only whole number bids are possible. Many students did not explain why the maximum does not exist. A key step is saying that the supremum is not contained inside the set.

Answer. The range is $\pi(\mathbb{R}) = [0, 10)$. The supremum of possible profits is $\sup \pi(\mathbb{R}) = \sup[0, 10) = 10$. But there is no maximum of the set $[0, 10)$, because the supremum does not lie within the set.

(v) (medium) Suppose that (X, d) is unbounded. Prove that there is a continuous function $f : X \rightarrow \mathbb{R}$ that does not have a maximum.

Comment. This sample solution is a proof by construction.

Answer. Pick any point $x^* \in X$. Let $f(x) = d(x, x^*)$. Recall that f is continuous. Moreover, f is unbounded, and thus has no maximum.

(vi) (medium) Suppose (X, d) is a disconnected metric space. Prove that there is a continuous function $f : X \rightarrow X$ that does not have any fixed point, i.e. there is no $x^* \in X$ with $f(x^*) = x^*$.

Comment. This sample solution is a proof by construction.

Answer. Since (X, d) is disconnected, there is a set A that is both open and closed. Moreover, its complement $B = X \setminus A$ is also both open and closed. Pick any $a^* \in A$ and any $b^* \in B$, and consider the function

$$f(x) = \begin{cases} a^* & \text{if } x \in B, \\ b^* & \text{if } x \in A. \end{cases}$$

Thus, $f(A) \subseteq B$ and $f(B) \subseteq A$, so there is no fixed point.

- (vii) (medium) Consider a contraction $f : X \rightarrow X$ of degree k on the metric space (X, d) . Let $A_0 = B_{r_0}(x_0)$ be an open ball, and let $A_{n+1} = f(A_n)$. Prove that A_n is contained in a ball of radius $r_n = r_0 k^n$.

Comment. This sample solution is a proof by construction.

Answer. Suppose the statement is true for n , i.e. that there exists some x_n such that $f^n(A_0) \subseteq B_{r_n}(x_n)$. Let $x_{n+1} = f(x_n)$ and notice that $r_{n+1} = kr_n$. We must prove that $f^{n+1}(A_0) \subseteq B_{r_{n+1}}(x_{n+1})$. Pick any $a_{n+1} \in f^{n+1}(A_0)$. There must be some $a_n \in f^n(A_0)$ such that $a_{n+1} = f(a_n)$. Since $f^n(A_0) \subseteq B_{r_n}(x_n)$, we deduce that $d(a_n, x_n) < r_n$. Since f is a contraction, we deduce that $d(f(a_n), f(x_n)) = d(a_{n+1}, x_{n+1}) < kr_n = r_{n+1}$. We deduce that $a_{n+1} \in B_{r_{n+1}}(x_{n+1})$ and conclude that $A_{n+1} \subseteq B_{r_{n+1}}(x_{n+1})$, as required.

- (viii) (hard) Suppose that $A \subseteq U$ and $B \subseteq V$ are non-empty sets, and U and V are disjoint open sets, and all four sets lie inside the metric space (X, d) . Prove that $A \cup B$ is disconnected.

Answer. We will use this lemma twice: Suppose (X, d) is a metric space and $A, Y \subseteq X$. If A is open inside (X, d) then $A \cap Y$ is open inside (Y, d) .

Proof of the lemma: Pick any point $y \in A \cap Y$. We need to prove y is an interior point of $A \cap Y$. Since $y \in A$ and A is an open set inside (X, d) , there is an open ball $B_r(y; X) \subseteq A$, where $B_r(y; X) = \{x \in X : d(x, y) < r\}$. Now consider the open ball $B_r(y; Y) = B_r(y; X) \cap Y$. We deduce that $B_r(y; Y) = B_r(y; X) \cap Y \subseteq A \cap Y$, as required.

Now, back to the question.

Let $Y = U \cup V$. By the lemma, U and V are open sets inside (Y, d) . Since they are complements of each other, U and V are also closed in (Y, d) . Thus (Y, d) is a disconnected metric space.

Now consider $Z = A \cup B$. By the lemma, $U \cap Z$ and $V \cap Z$ are open sets inside (Z, d) . Since U and V are disjoint and $B \subseteq V$, we deduce that $U \cap Z = U \cap (A \cup B) = (U \cap A) \cup (U \cap B) = U \cap A = A$. Similarly, $V \cap Z = B$. Thus, we have established that A and B are open sets inside (Z, d) . Their complements are closed, so A and B are both open and closed sets. We deduce that (Z, d) is a disconnected metric space.

54: AME, May 2024

Part A

It is likely that Scotland will be largely unaffected by global warming, whereas agriculture will become more difficult in Spain. Assume that food is produced from land and labour. Assume that productivity declines as temperatures rise, but that each degree increase has a smaller impact on productivity. Households in each country own the land and the local farms, and they supply labour and land and consume food.

- (i) Write down a competitive model of the land and labour markets in each country, and the global food market.

Answer.

Households. In country $c \in \{E, S\}$ (“España” and “Scotland”), there are n_c households. Each household is endowed with ℓ_c units of land which it supplies inelastically at price r_c , 1 unit of time of which it chooses h_c to work w_c , and buys a_c units of food at price p . It also receives portion of the profits of the local farm, π_c . This gives it utility $u(h_c, a_c)$. The household’s utility maximisation problem is

$$\begin{aligned} \max_{h_c, a_c} u(h_c, a_c) \\ \text{s.t. } pa_c = w_ch_c + \ell_cr_c + \frac{\pi_c}{n_c}. \end{aligned}$$

Farms. The farm in country c chooses how much labour H_c to hire, how much land to use L_c , and is affected by the local temperature t_c . It produces $f(H_c, L_c, t_c)$, where f is differentiable, and it is increasing in (H_c, L_c) , but decreasing and convex in t_c . Its profit maximisation problem is

$$\pi_c(p; w_c, r_c; t_c) = \max_{H_c, L_c} pf(H_c, L_c, t_c) - w_cH_c - r_cL_c.$$

Equilibrium. Prices (p, w_E, w_S, r_E, r_S) and quantities $(h_E, h_S, L_E, L_S, a_E, a_S)$ constitute an equilibrium if the quantities are optimal choices in their respective problems above, and all markets clear:

$$\begin{aligned} n_E a_E + n_S a_S &= f(H_E, L_E, t_E) + f(H_S, L_S, t_S) \\ n_E h_E &= H_E \\ n_S h_S &= H_S \\ n_E \ell_E &= L_E \\ n_S \ell_S &= L_S. \end{aligned}$$

- (ii) Assuming prices do not change, prove that as Spanish temperatures increase,

- (a) Spanish farms’ profits decline, and

Answer. Suppose that (H_E^*, L_E^*) are optimal choices when faced with $(p; w_E, r_E; t_E)$. By the envelope theorem

$$\begin{aligned} \frac{\partial \pi_E(p; w_E, r_E; t_E)}{\partial t_E} &= \frac{\partial}{\partial t_E} (pf(H_E^*, L_E^*, t_E) - w_E H_E^* - r_E L_E^*) \\ &= p \frac{\partial}{\partial t_E} f(H_E^*, L_E^*, t_E). \end{aligned}$$

Now, $p > 0$, and the production function f is assumed to be decreasing in t_E , so the right side is a negative number. We conclude that the firm's profits are decreasing as Spanish temperatures rise.

- (b) Spanish farms' marginal profits increase, i.e. the derivative of profits with respect to temperature increases.

Answer. For each possible choice (H_E, L_E) , the Spanish farm's profits

$$pf(H_E, L_E, t_E) - w_E H_E - r_E L_E$$

is a convex function of Spanish temperature t_E . The profit function π_E is the upper envelope of convex functions of t_E , so we deduce that π_E is convex in t_E . This means implies that $\frac{\partial \pi_E}{\partial t_E}$ is an increasing function of t_E .

Comment. The marginal profit is negative, since global warming harms farm output. But as temperatures rise, the farm reduces its exposure to global warming (by reducing production), and the underlying production function assumes that subsequent temperature rises have less impact than initial ones. Putting these two things together gives the conclusion that profits decrease more after initial temperature rises than subsequent rises.

- (iii) To mitigate global warming risks, the Spanish farms decide to merge with the Scottish farms. Write down a profit function for the conglomerate.

Answer. The profit function is just the sum of the profit functions of the divisions.

$$\pi(p; w_E, r_E, w_S, r_S; t_E, t_S) = \pi_E(p; w_E, r_E; t_E) + \pi_S(p; w_S, r_S; t_S).$$

Part B

- (i) (easy) Give a counterexample to the following false claim. Let A and B be subsets of a metric space (X, d) such that $A \cup B$ is connected. Then $A \cap B$ is connected.

Answer. Let $A = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{1\})$ be the left and top edges of a square, and let $B = (\{1\} \times [0, 1]) \cup ([0, 1] \times \{0\})$ the bottom and right edges. The union (all edges) $(A \cup B, d_2)$ is connected, since intervals are connected, Cartesian products are connected, and unions of overlapping connected sets are connected. But the intersection $\{(0, 0), (1, 1)\}$ is disconnected.

- (ii) (easy) Give a counterexample to the following false claim. Let $f : X \rightarrow Y$ be a continuous map between metric spaces. If $W \subseteq Y$ is compact, then $f^{-1}(W) \subseteq X$ is compact.

Answer. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 0$, using d_2 to measure distances in both spaces. Now, $W = \{0\}$ is compact since it is finite. Yet $f^{-1}(W) = \mathbb{R}$, which is not compact.

- (iii) (easy) Give a counterexample to the following false claim. Suppose $f : X \rightarrow Y$ is a continuous function from (X, d_X) to (Y, d_Y) . If x_n is a Cauchy sequence in (X, d_X) , then $y_n = f(x_n)$ is a Cauchy sequence in (Y, d_Y) .

Answer. Consider (X, d_2) and (Y, d_2) , where $X = (0, 1)$ and $Y = \mathbb{R}_+$. Let $x_n = 1/n$ and $f(x) = 1/x$ so that $y_n = f(x_n) = n$. Notice that x_n is a Cauchy sequence, since x_n is convergent inside (\mathbb{R}, d_2) . However y_n is not a Cauchy sequence, since $d(x_n, x_m) = |n - m|$.

- (iv) (easy) Let (X, d) be a metric space and $f : X \rightarrow X$ be a self-map such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. (These functions are sometimes called isometries.) Give an example of (X, d) and f such that $f(X) \neq X$.

Answer. Consider $(X, d) = (\mathbb{R}_+, d_2)$ and $f(x) = x + 1$. Then $f(X) = [1, \infty) \neq X$.

- (v) (medium) Consider any compact metric space (X, d) . Prove that if $A \subseteq X$ is not compact, then there exists an unbounded continuous function $f : A \rightarrow \mathbb{R}$.

Answer. Recall that closed subsets of compact spaces are compact. We deduce that A is not closed. So there is a sequence $a_n \in A$ that converges to some point $x \in X \setminus A$. Let $f(a) = 1/d(a, x)$. First, $f : A \rightarrow \mathbb{R}$ is well-defined, since $d(a, x) > 0$ for all $a \in A$. Second, f is continuous, since it is the composition of two continuous functions d , and the reciprocal function. Third, f is unbounded since $f(a_n) \rightarrow \infty$.

- (vi) (medium) Prove that the closed ball $\{f \in C[0, 1] : d_\infty(f, 0) \leq 1\}$ is not compact inside $(C[0, 1], d_\infty)$.

Answer. Consider the sequence of functions $f_n(x) = \min\{1, d(x, [n, \infty))\}$, where $d(x, A)$ is defined in the usual way as $\inf_{a \in A} d(x, a)$. Now, $d_\infty(f_n, f_m) = 1$ for all $n \neq m$. (To see this, suppose $n < m$. Then $d_\infty(f_n, f_m) = d(f_n(n), f_m(n)) = d(1, 0) = 1$.) So f_n does not have any Cauchy subsequences, and therefore has no convergent subsequences.

- (vii) (medium) An ecologist is looking for Tasmanian pygmy possums in a forest. Each day, she decides whether to continue the search, or to give up. She has a (probabilistic) belief p that the forest contains possums. It costs c to spend a day looking, and if the forest does have any possums, the chance of finding one on any day is q . The reward for finding a possum is 1. Her discount rate is β , and her Bellman equation is

$$V(p) = \max\{0, -c + pq + (1 - pq)\beta V(p')\},$$

where $p' = \frac{p(1-q)}{p(1-q)+(1-p)}$ is the posterior belief obtained by Bayes' rule.

Write down the Bellman operator and prove that it is a contraction. *Hint: don't forget to write the domain of the operator.*

Answer. The Bellman operator $F : B[0, 1] \rightarrow B[0, 1]$ is

$$F(V)(p) = \max\{0, -c + pq + (1 - pq)\beta V(p')\},$$

where $p' = \frac{p(1-q)}{p(1-q)+(1-p)}$.

I will adapt Blackwell's Lemma to show that F is a contraction of degree β on the

space $(B[0, 1], d_\infty)$. Pick any functions $V, W \in B[0, 1]$. Then

$$\begin{aligned}
F(V)(p) &= \max \{0, -c + pq + (1 - pq)\beta V(p')\} \\
&= \max \{0, -c + pq + (1 - pq)\beta[V(p') + W(p') - W(p')]\} \\
&\leq \max \{0, -c + pq + (1 - pq)\beta W(p')\} + (1 - pq)\beta \max \{0, V(p') - W(p')\} \\
&\leq \max \{0, -c + pq + (1 - pq)\beta W(p')\} + \beta \max \{0, V(p') - W(p')\} \\
&= F(W)(p) + \beta \max \{0, V(p') - W(p')\} \\
&= F(W)(p) + \beta d_\infty(V, W)
\end{aligned}$$

so we deduce

$$F(V)(p) - F(W)(p) \leq \beta d_\infty(V, W).$$

Swapping the roles of V and W gives

$$F(W)(p) - F(V)(p) \leq \beta d_\infty(V, W).$$

We deduce that

$$|F(V)(p) - F(W)(p)| \leq \beta d_\infty(V, W).$$

Since the inequality holds for all p , we deduce that

$$d_\infty(F(V), F(W)) \leq \beta d_\infty(V, W),$$

and conclude that F is a contraction of degree β .

- (viii) (hard) Let A and B be closed subsets of (X, d) such that $A \cup B$ and $A \cap B$ are connected. Prove that A and B are connected. *Hint: If A were disconnected, then $A \cap B$ would lie entirely in one of two pieces of A .*

Answer. Suppose for the sake of contradiction that A is disconnected. Then there are two disjoint closed sets U and V inside (X, d) such that $A = (A \cap U) \cup (A \cap V)$ and both $A \cap U$ and $A \cap V$ are non-empty.

Since $A \cap B$ is connected, $A \cap B$ lies entirely inside either U or V . Without loss of generality, assume it's the former, i.e. $A \cap B \subseteq U$.

Now consider formulating $A \cup B$ as the union of two sets, $A \cap V$ and $(A \cap U) \cup B$.

Both sets are non-empty, since $A \cap U$ and $A \cap V$ are non-empty. Both sets are closed, since all three ingredients, $A \cap V$, $A \cap U$ and B are closed. The sets are disjoint, since $A \cap V$ is disjoint from both $A \cap U$ and B . So $A \cup B$ is disconnected, a contradiction.

55: Micro 1, May 2024

Suppose there are two types of smartphone users: gamer and social. They have different preferences about the processor speed and storage size, and benefit from at most one phone. There are two smartphone makers that make one type of phone each. They can choose any design they like, and have access to the same technology. The households supply labour to the makers, hold equal shares in both, and buy phones.

- (i) Write down a competitive model of the phone markets (two of which are active) and the labour market.

Answer.

Households. There are n households. Of these, there are n_t households of type $t \in \{G, S\}$, gamer or social. Each household of type t chooses a phone $(z_t, s_t) \in \mathbb{R}_+^2$ with speed z_t and storage s_t at price $p(z_t, s_t)$. It buys $q_t \in \{0, 1\}$ of them. The household supplies labour ℓ_t at a wage of w , where $\ell_t = 1$ denotes full time work. The household receives dividends $(\pi_1 + \pi_2)/n$. This gives them a utility of $u(q_t, z_t, s_t, \ell_t)$. The utility maximisation problem is

$$\begin{aligned} \max_{q_t, z_t, s_t, \ell_t} & u(q_t, z_t, s_t, \ell_t) \\ \text{s.t.} & p(z_t, s_t)q_t = w\ell_t + \frac{\pi_1 + \pi_2}{n}. \end{aligned}$$

Firms. Firm $i \in \{1, 2\}$ chooses which design (Z_i, S_i) to make, and how many workers L_i to hire, and produces $f(Z_i, S_i, L_i)$ phones. The profit function is

$$\pi_i(p; w) = \max_{Z_i, S_i, L_i} p(Z_i, S_i)f(Z_i, S_i, L_i) - wL_i.$$

Equilibrium. Prices (p, w) and quantities (q_t, z_t, s_t, ℓ_t) and (Z_i, S_i, L_i) form an equilibrium if the quantities solve the problems they appear in above, and all markets clear, i.e.

$$\begin{aligned} n_G \ell_G + n_S \ell_S &= L_1 + L_2 \\ \sum_{t \in \{G, S\}} n_t q_t I(z_t = z, s_t = s) &= \sum_{i \in \{1, 2\}} f(Z_i, S_i, L_i) I(Z_i = z, S_i = s) \text{ for all } (z, s) \in \mathbb{R}_+^2. \end{aligned}$$

- (ii) Suppose the smartphone makers merge. Write down the profit function of the conglomerate.

Answer. $\pi(p; w) = \pi_1(p; w) + \pi_2(p; w)$.

- (iii) Suppose the price of the phone type sold by one of the firms increases (and all other prices stay the same). How would both firms' choices be affected?

Answer. Suppose that at prices (p^*, w^*) , that firm 1 chooses (Z^*, S^*, L^*) . If $p(Z^*, S^*)$ increases, then the firm would not change the design, as the (Z^*, S^*) is now more profitable, and the other designs is unaffected.

By the envelope theorem,

$$\begin{aligned}\frac{\partial \pi_1(p; w)}{\partial p(Z^*, S^*)} &= \frac{\partial}{\partial p(Z^*, S^*)} \left\{ p(Z^*, S^*) f(Z^*, S^*, L) - wL \right\} \Big|_{L=L^*} \\ &= f(Z^*, S^*, L) \Big|_{L=L^*} \\ &= f(Z^*, S^*, L^*).\end{aligned}$$

Notice that π_1 is the upper envelope of linear functions, one for each choice (Z, S, L) . So π_1 is a convex. Thus, the left side of the envelope formula is increasing. We deduce that the right side is increasing, and conclude that firm 1's supply of firms is increasing in the price of the phones.

Turning attention to firm 2, notice that the firms have equal profits, i.e. $\pi_1(p^*, w^*) = \pi_2(p^*, w^*)$. Otherwise one of the firm with a lower profit could copy the choices of the other firm, and increase their profits. When $p(Z^*, S^*)$ increases, firm 1's profit increases. Firm 2 can also increase its profit by copying firm 1. To conclude: firm 2 would react to the price increase by copying firm 1's choices.

- (iv) Prove that if there is excess demand for all types of phones, then there is excess supply of labour. For full credit, adapt all the proofs from class that you need.

Answer. This follows from Walras law.

Summing up the budget constraints gives

$$\begin{aligned}\sum_{t \in \{G, S\}} n_t \left[p(z_t, s_t) q_t - w l_t - \frac{\pi_1 + \pi_2}{n} \right] &= 0 \\ \sum_{t \in \{G, S\}} n_t [p(z_t, s_t) q_t - w l_t] - \pi_1 - \pi_2 &= 0.\end{aligned}$$

Substituting in the firms' profits gives

$$\sum_{t \in \{G, S\}} n_t [p(z_t, s_t) q_t - w l_t] - \sum_{i \in \{1, 2\}} [p(Z_i, S_i) f(Z_i, S_i, L_i) - w L_i] = 0.$$

Since there is excess demand for all types of phones, we deduce

$$\begin{aligned}\sum_{t \in \{G, S\}} n_t [-w l_t] + w L_1 + w L_2 &\leq 0 \\ - \sum_{t \in \{G, S\}} n_t l_t + L_1 + L_2 &\leq 0. \\ L_1 + L_2 &\leq \sum_{t \in \{G, S\}} n_t l_t.\end{aligned}$$

We conclude that there is excess supply of labour.

- (v) Suppose the 5G Coalition wants to convince the government to reduce smartphone production. Devise a lump-sum tax policy that minimises the number of phones produced.

Answer. I will devise a policy that results in only one phone being produced.

Pick household 1's favourite deluxe design (\hat{Z}, \hat{S}) from the set of designs that require $n - 1$ full-time workers to produce a single phone. Consider the allocation of resources in which all households except household 1 work full time and receive no phones, and household 1 does not work, and receives a single (\hat{Z}, \hat{S}) phone. Either this allocation is efficient, or there is another one that dominates it in which household 1 works, and household 1 receives an even better phone. There is no way to make the other households better off without making household 1 worse off.

By the second welfare theorem, it is possible to design lump sum transfers to implement this allocation. Specifically, if (p^*, w^*) are equilibrium prices when the players are endowed with this efficient allocation, then taxing all households except household 1 by $w^* + (\pi_1 + \pi_2)/n$, and giving household 1 a lump sum of $(n - 1)w^* + \frac{n-1}{n}(\pi_1 + \pi_2)$ would implement the allocation.