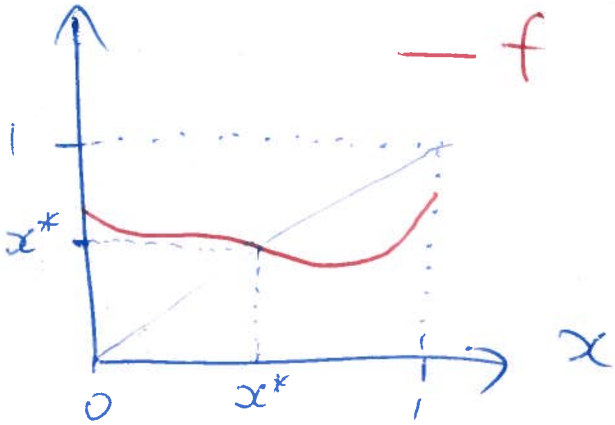
Existence of Equilibrium (4.6)

Theorem Consider a pure-exchange economy in which each utility function $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ is continuous, strictly increasing (or locally non-satiated, i.e. every open ~~ball~~ neighbourhood contains a better choice) and strictly quasi-concave and aggregate endowments are strictly positive (i.e. $\sum_h e_{hn} > 0$ for all n). Then there exists a pure-exchange equilibrium (x^*, p^*).

Detour: fixed points. Let $f : X \to X$. A fixed point of f is a point $x^* \in X$ such that $x^* = f(x^*)$.

Theorem 4.5 (A ~~very~~ too-simple fixed point theorem)

If $f : [0,1] \to [0,1]$ is continuous, then f has a fixed point.



Proof Consider $g(x) = f(x) - x$.
Notice that $g(0) \geq 0$ and $g(1) \leq 0$ and g is continuous.
By the Intermediate Value Theorem, g crosses 0 at least once, i.e. there is some $x^* \in [0,1]$ such that $g(x^*) = 0 \iff f(x^*) - x^* = 0$
$\iff f(x^*) = x^*$. $\square$

A better theorem:

Theorem (Brouwer's fixed point theorem)

If $f: X \to X$ is continuous function and $X \subset \mathbb{R}^N$ is non-empty, convex and compact, then f has a fixed point.

Back to proving existence of equilibrium

Recall that (x^*, p^*) is an equilibrium if and only if x^* is an optimal choice for each household given p^* and $\underbrace{z(p^*)}_{\text{excess demand}} = \overbrace{0}^{\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}}$.

Since ~~...~~ there is always an optimal choice given p^* and $p_n^* > 0$ (under the assumptions on u_h), the problem boils down to:

Is there a $p^* \in \mathbb{R}_+^N$ such that $z(p^*) = 0$?

First problem: What if $p_n^* = 0$?

$\bar{z}_n(p) = \min\{1, z_n(p)\}$

truncated excess demand ≤ 1.

First proposal:

$p_n' = p_n + \bar{z}_n(p)$?

Problem: what about excess supply? Could have $\bar{z}_n(p) = -\infty$.

Second ~~pr~~ proposal:

$$p_n'' = p_n + \underbrace{\max\{0, z_n(p)\}}$$

never cut prices.
Prices only increase.

Seems crazy: even if there are lots of vacant, the price of hotel rooms stays the same!

But by Walras law, if there is excess supply of hotel rooms, there is excess demand of something else ~~t~~ – e.g. taxis. So at least one price will increase.

~~Third prop~~ Problem: To use Brouwer's fixed point theorem, the compactness assumption means prices must lie in [0, 1].

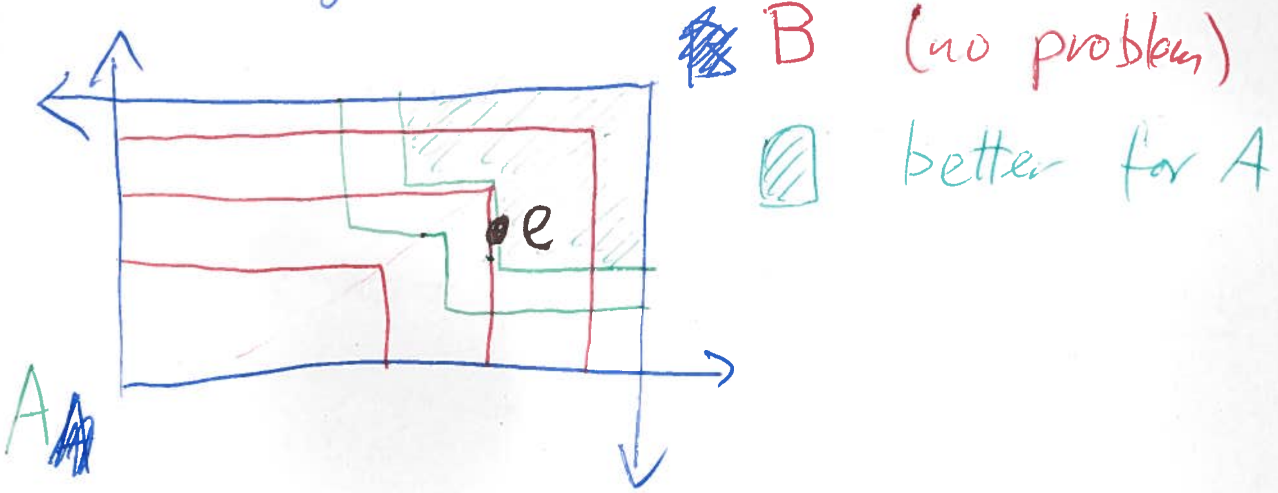
Third proposal

$$p_n''' = \frac{p_n''}{\sum_{m=1} p_m''} \quad \text{so that } \sum_n p_n''' = 1.$$

Note: $z(p''') = z(p'')$ because prices are relative.

So the function $f: p \mapsto p'''$ *turns p into p''', as defined above*
is continuous,
with domain and co-domain
$$X = \{p \in [0,1]^n : \sum_{n=1}^{N} p_n = 1\}$$
being a non-empty, convex and compact set.
So Brouwer's fixed point theorem says that f has a fixed point p^*, which by the logic in the second proposal has $z(p^*) = 0$,
So p^* is an equilibrium price vector. $\square$

If we drop quasi-concavity, then we might get this situation



4.7 Implementation

Def Consider a pure-exchange ~~equilibrium~~ economy in N goods involving utility functions u_h and endowments e_h. A feasible allocation x^* along with prices p^* form a <u>pure-exchange equilibrium with lump-sum taxes $t_h \in \mathbb{R}$</u> if

① we require $\sum_h t_h = 0$,

② $$x_h^* \in \underset{\hat{x}_h \in \mathbb{R}^N_+}{\arg\max}\; u(\hat{x}_h) \quad \text{s.t. } p^* \cdot \hat{x}_h \leq p^* \cdot e_h - t_h$$
for all h

③ markets clear: $\sum_h x_h^* = \sum_h e_h$.

Theorem (Second welfare theorem)

Consider a pure exchange economy (u, e). Suppose ~~each~~ each u_h is continuous, increasing and strictly quasi-concave and $\sum_h e_{hn} > 0$ for all n. If $x^* \in \mathbb{R}^{NH}_{++}$ is an efficient allocation, then there exist~~s~~ prices $p^* \in \mathbb{R}^N_+$ and taxes t^* such that (x^*, p^*, t^*) form an equilibrium.

Proof (based on Maskin & Roberts, 1980)

Pretend endowments are x^* (not e^*). By the existence theorem, there exists an equilibrium (x^{**}, p^*) given endowments x^*. We assumed x^* is efficient. I claim all households h are indifferent between x^*_h and x^{**}_h. Since refusing to trade is feasible for all households, $u(x^{**}_h) \geq u(x^*_h)$. Since x^{**} ~~are~~ is feasible and x^* is efficient, then, if any household $\tilde{h}$ with $u(x^{**}_{\hat{h}}) > u(x^*_{\hat{h}}$ would imply another household is worse off (contradiction).

Therefore x^*_h is an optimal choice given endowment x^*_h and prices p^*. We conclude (x^*, p^*) is an equilibrium with endowment x^*

We now construct the taxes t_i^* as follows:

* before Robin Hood, (x^*) budget constraint was:

$$p^* \cdot \hat{x}_h \leq p^* \cdot e_h$$

* after Robin Hood:

$$p^* \cdot \hat{x}_h \leq p^* \cdot x_h^*$$

* difference: $t_h^* = p^* \cdot (e_h - x_h^*)$
* new budget constraint: (Margaret Thatcher)

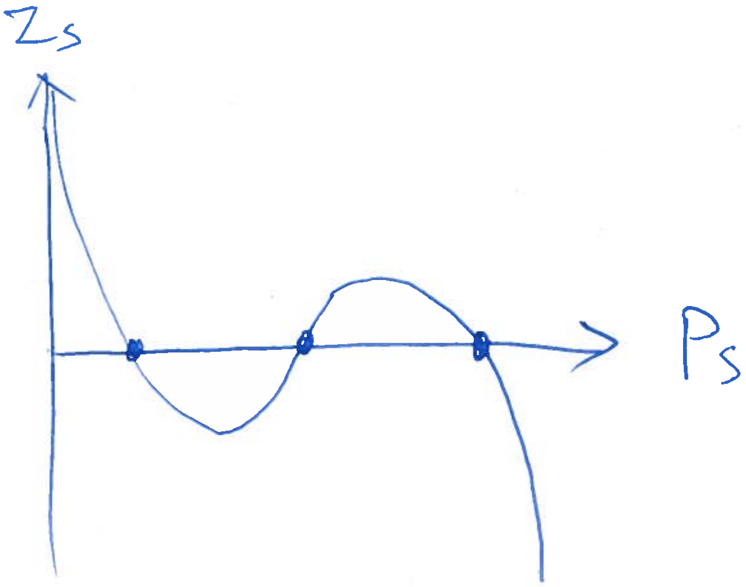
$$p^* \cdot \hat{x}_h \leq p^* \cdot e_h - t_h^*.$$

Note: $\sum_h t_h^* = \sum_h p^* \cdot (e_h - x_h^*) = p^* \cdot \underbrace{\sum_h (e_h - x_h^*)}_{0 \text{ by market clearing}}$

$$= 0.$$

Therefore, (x^*, p^*, t^*) is an equilibrium ~~[illegible]~~ with endowment e

$\square$



Excess demand