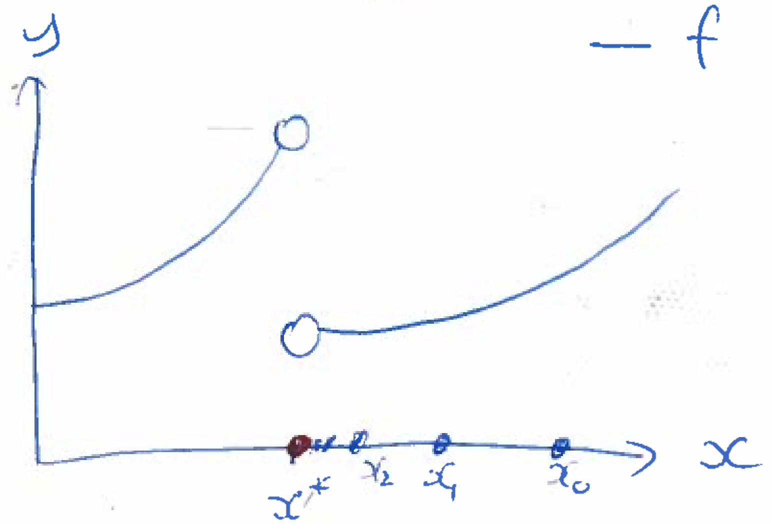
C6 Continuity

Spot the mistake



$x_n \to x^*$

but $f(x_n) \not\to f(x^*)$

$\Rightarrow$ f is discontinous.

<u>Problem:</u> We did not specify f properly. We need to say $f: X \to Y$, along with metrics d_X and d_Y.

Possibility:

① $X = \mathbb{R}$, and $Y = \mathbb{R}$, $d_X = d_Y = d_2$.
But $f(x^*)$ is not defined, so $\text{domain}(f) \neq \mathbb{R}$. <u>Ruled out.</u>

② $X = \mathbb{R} \setminus \{x^*\}$. Then $x_n \not\to x^*$.
If we set $Y = \mathbb{R}$ and $d_X = d_Y = d_2$ then f is continuous.

C7 Completeness

$(X, d) = ((0,1], d_1)$ has a "hole".
$x_n = \frac{1}{n}$ "wants" to converge.
But since $0 \notin X$, it does not converge.

<u>Def</u> Let (X, d) be a metric space.
A sequence $x_n \in X$ is called
a <u>Cauchy sequence</u> ("wants to converge") if for every radius
$r > 0$, there exists a number N
such that

$$d(x_n, x_m) < r \quad \text{for all } n, m > N.$$

no x^* here!

<u>Def</u> (X, d) is <u>complete</u> if
every Cauchy sequence $x_n \in X$ is
a convergent sequence.

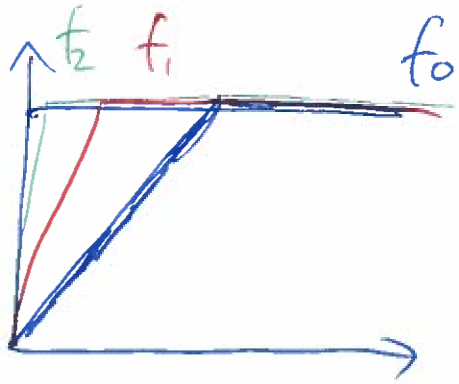
<u>Examples</u>:
* $(\mathbb{R}, d_2)$ is complete
* $((0,1], d_1)$ is not complete.
* $(\mathbb{Q}, d_1)$ is not complete – all irrational numbers can be reached as the limit of a sequence of rational numbers in

$(\mathbb{R}, d_1)$.

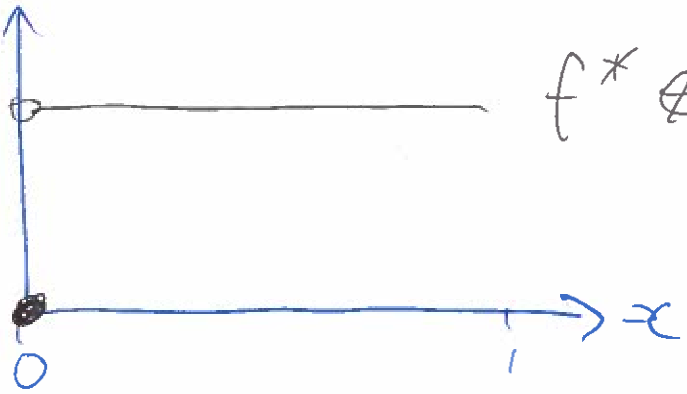
* $(CB[0,1], d_1)$ is <u>NOT</u> complete

$\{ f : [0,1] \to \mathbb{R}, \ f$ is continuous and bounded$\}$,

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$



wants to converge to



$f^* \notin CB[0,1]$.

<u>Theorem</u> If $x_n \in X$ is a convergent sequence in (X, d), then x_n is a Cauchy sequence.

<u>Theorem</u> If $x_n \in X$ is a Cauchy sequence in (X,d) and $y_n \to y^*$ is a ~~sub~~ convergent subsequence of x_n, then $x_n \to y^*$.

<u>Idea</u>: Cauchy $\Rightarrow$ wants to converge

$y_n \to y^* \Rightarrow$ no hole.

<u>Theorem</u> If $x_n \in X$ is a Cauchy sequence in (X,d) ~~and~~ then x_n is bounded.

<u>Theorem</u> If $x_n \in X$ is a Cauchy sequence in (X,d) and y_n is a subsequence of x_n, then y_n is a Cauchy sequence.

<u>Theorem</u> Let (X,d_X) and (Y,d_Y) be metric spaces. If (Y,d_Y) is ~~a~~ complete, then $(B(X,Y), d_\infty)$ $\{f: X \to Y : f \text{ is bounded}\}$ and $(CB(X,Y), d_\infty)$ are complete metric spaces.

18 Fixed Points

Def A function f is a self-map if $f: X \to X$.

Def Let $f: X \to X$ [be a self-map]. A point $x^* \in X$ is a fixed point if $x^* = f(x^*)$.

Def Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is Lipschitz continuous of degree a if for every $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq a\, d_X(x, x').$$

Def Let (X, d) be a metric space. The self-map $f: X \to X$ is a contraction if it is Lipschitz continous of degree $a < 1$.

i.e. $d(f(x), f(x')) \leq a\, d(x, x') < d(x, x')$

__Theorem__ (Banach's fixed point theorem) Let (X, d) be a complete metric space. If $f: X \to X$ is a contraction of degree a, then

(i) f has a unique fixed point x^*.

("one and only one" or "the")

(ii) Given any $x_0 \in X$, the sequence $x_{n+1} = f(x_n)$ converges to x^*.

(iii) $d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1)$.

$= f(x_0)$

__Proof__ __Uniqueness:__ Suppose x^* and x^{**} are fixed points of f. this implies $d(f(x^*), f(x^{**})) = d(x^*, x^{**})$.

But $d(f(x^*), f(x^{**})) \leq a\, d(x^*, x^{**})$, by the contraction property. This implies $d(x^*, x^{**}) = 0$, i.e. $x^* = x^{**}$.

Or as a proof by contradiction, if $x^* \neq x^{**}$, then:

$$\cancel{d(x^*, x^{**})} \leq a\, \cancel{d(x^*, x^{**})}$$

$$1 \leq a. \quad \text{But } a < 1.$$