Theorem C1 Let x_n be a sequence in a metric space (X,d). If x_n is unbounded then x_n does not converge.

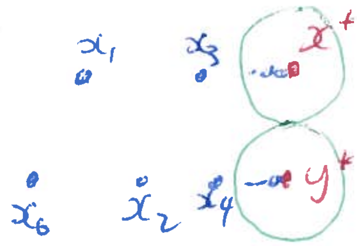
x_n is bounded if there is a radius $r>0$ such that $d(x_0, x_n) < r$ for all n.

Theorem C2 A sequence x_n in (X,d) can converge to at most one point in X.

Proof Suppose for the sake of contradiction that $x_n \to x^*$ and $x_n \to y^*$ and $x^* \neq y^*$.

~~Since $x_n \to x^*$~~ Let $r = \frac{1}{2} d(x^*, y^*)$.

Since $x_n \to x^*$, there exists N_x such that $d(x_n, x^*) < r$ for all $n \geq N_x$.



Similarly, there is a number N_y for y^*.

Let $N = \max\{N_x, N_y\}$.

Then $d(x_N, x^*) < r$ and $d(x_N, y^*) < r$.

By the triangle inequality,

$$\begin{aligned} d(x^*, y^*) &\leq d(x^*, x_N) + d(x_N, y^*) \\ &< r + d(x_N, y^*) \\ &< r + r \\ &= 2r = d(x^*, y^*). \end{aligned}$$

↯

Def We say that y_n is a subsequence of x_n if there exists a strictly increasing sequence $k_n \in \mathbb{N}$ (i.e with $k_{n+1} > k_n$) such that $y_n = x_{k_n}$.

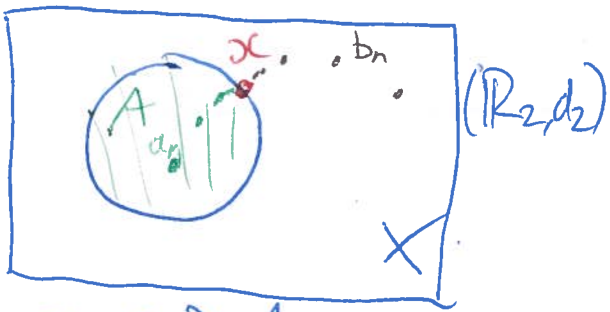
Theorem C3 If $x_n \to x^*$ and y_n is a subsequence of x_n, then $y_n \to x^*$.

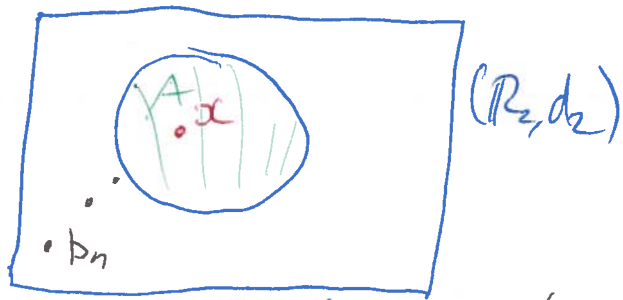
C3 Boundaries

Def Let A be any subset of a metric space (X, d). A point $x \in X$ is a <u>boundary point</u> of A if:

(i) there exists a sequence $a_n \in A$ such that $a_n \to x$, and ("x is almost in A")

(ii) there exists a sequence $b_n \in X \setminus A$ such that $b_n \to x$. ("x is almost outside of A")





$x \in \partial A$ $\qquad\qquad$ $x \notin \partial A$ $\quad b_n \not\to x$

The set of boundary points of A is called the <u>boundary of A</u>, written ∂A.

Examples:

* In $(\mathbb{R}, d_2)$, $\partial[0,1] = \{0, 1\}$.

  $a_n = 0$ and $b_n = -\frac{1}{n}$

* In $(\mathbb{R}, d_2)$, $\partial(0,1) = \{0, 1\}$

  $a_n = \frac{1}{n+2}$ and $b_n = 0$

* In $(\overbrace{[0,1]}^{X}, d_2)$, $\partial \underbrace{[0,1]}_{A} = \emptyset$.

  $0 \notin \partial[0,1]$ because there is no sequence $b_n \in X \setminus A$ (s.t. $b_n \to 0$)

* In $([0,1], d_2)$, $\partial(0,1) = \{0, 1\}$

* In $(\underbrace{\mathbb{R}_+}_{[0,\infty)}, d_2)$, $\partial[0,1] = \{1\}$.

* In $(\mathbb{R}_+, d_2)$, $\partial(0,1) = \{0, 1\}$.

C4 Closed sets

Def Suppose A is a set in (X, d). We say A is a <u>closed set</u> if there is no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$.

Examples:

* In $(\mathbb{R}, d_2)$, $[0,1]$ is a closed set.
  eg: $a_n = \frac{1}{n+2} \to 0$ does not "escape" from $[0,1]$.
* In (X,d), both $\emptyset$ and X are closed sets.
* $(0,1)$ <u>is</u> a closed set inside $((0,1), d_2)$ but <u>not</u> inside $(\mathbb{R}, d_2)$.

Theorem C4 Suppose A is a subset of (X,d). Then A is closed if and only if A contains its boundary, i.e. $\partial A \subseteq A$.

C5 Open sets

Def Consider a metric space (X, d). Then <u>open ball</u> centred at x with radius $r > 0$ is $N_r(x) = \{y \in X : d(x,y) <$

Def We say $x \in A$ is an interior point of A if there is an open ball $N_r(x)$ such that $N_r(x) \subseteq A$.
The set of interior points of A is called the interior of A.
If A equals its interior, then we say A is an open set.
If A is an open set and $x \in A$, then we say A is an open neighbourhood of x.