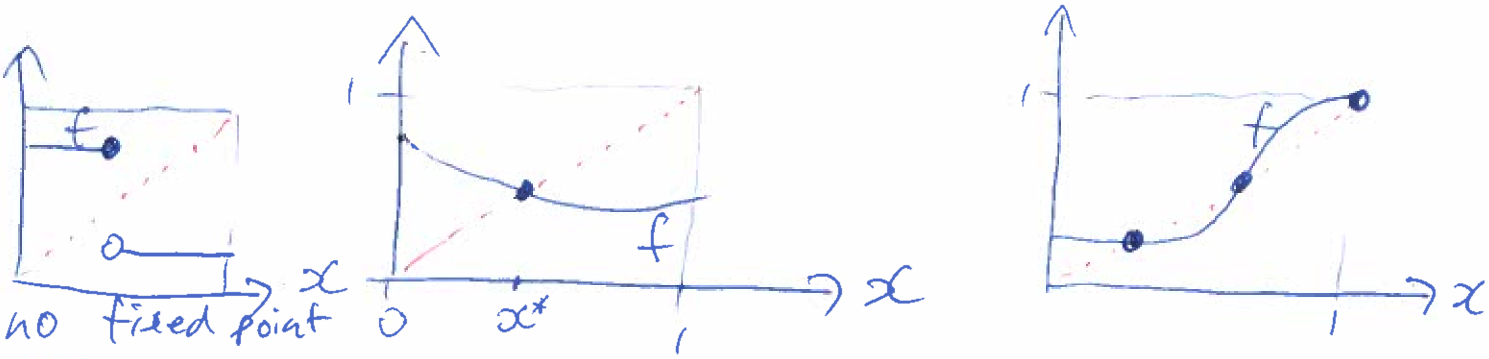
4.6 Existence of Equilibrium (*)

We will use a fixed point theorem.

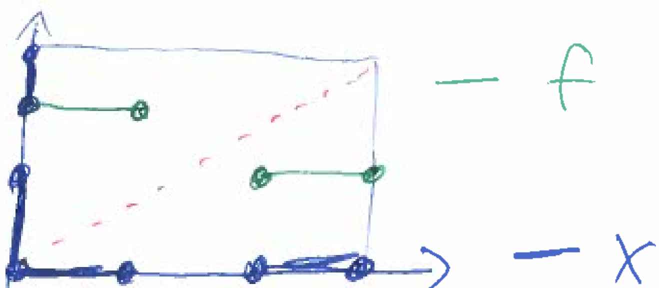
Theorem If $f:[0,1] \to [0,1]$ is continuous function, then f has a fixed point, i.e. x^* such that $x^* = f(x^*)$.



Brouwer's Fixed Point Theorem

If $f: X \to X$ is continuous, and $X \subset \mathbb{R}^N$ and X is non-empty, convex, closed, and bounded, then f has a fixed point.

e.g. If $X = \mathbb{R}$ ← unbounded, and $f(x) = x+1$, then f has no fixed point.



f is continuous, but X is not convex.

Theorem 4.5 Consider a pure-exchange economy (u, e) in which $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ is continuous, strictly increasing, & strictly quasi-concave and aggregate endowment are positive i.e. $\sum_{h \in H} e_{hn} > 0$ for all goods n.

In such an economy, there exists a pure-exchange equilibrium (x^*, p^*).

Proof ① Let $\bar{z}_i(p) = \min\{1, z_i(p)\}$ is a function by q-c. be the truncated excess demand function. Even though $z_i(p)$ might be infinite, $\bar{z} : \mathbb{R}^N_+ \to \mathbb{R}^N$.

Notice that p^* is an equilibrium price if and only if $\bar{z}(p^*) = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

② Define our adjusted prices as

$$p_i' = p_i + \underbrace{\max\{0, \bar{z}_i(p)\}}_{\geq 0,\ \text{so prices can only increase}}$$

By Walras law, $p' = p$ if and only if p is an equilibrium price.

③ Define $p_i'' = \dfrac{p_i'}{\sum_j p_j'}$. Eg if $p' = (\frac{1}{2}, \frac{1}{2}, 1)$. Then $p'' = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

④ Let $X = \{p \in \mathbb{R}^N_+ : \sum p_i = 1\}$.
$\ne \emptyset$, convex, closed, bounded

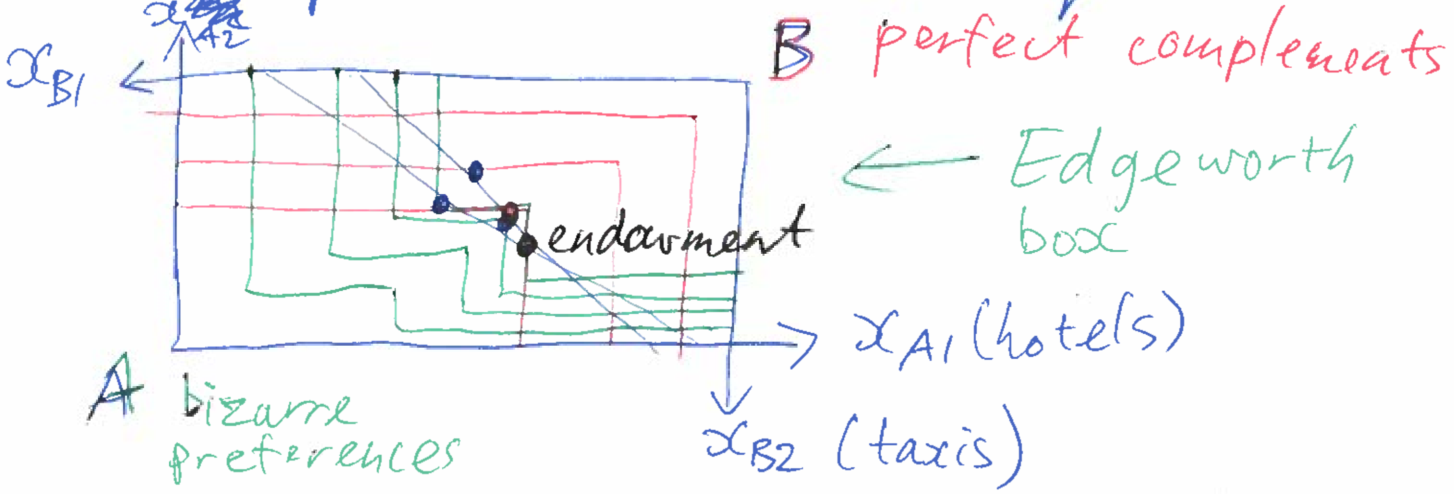
Let $f: p \mapsto p''$, as defined in steps ①–③.

In section 3.7, we proved that $x_h(p)$ is continuous, so f is continuous.

So by Brouwer's fixed point theorem, there exists some $p^* \in X$ such that $p^* = f(p^*)$.

~~Since~~ So $z(p^*) = 0$ (by the construction of f), so p^* is an equilibrium price. □

An example without an eq:



A always chooses a L-shaped corner, and B chooses a ⌉-shaped corner. So neither A nor B is dissatisfied with every allocation.

4.7 Implementation of Efficient Allocations — not bonus

Def Consider a pure-exchange economy (u, e). Then (x^*, p^*) is a pure-exchange equilibrium with lump-sum taxes t if

1. zero total taxes are levied, i.e. $\sum_{h \in H} t_h = 0$,
2. $$x_h^* \in \arg\max_{\hat{x}_h \in \mathbb{R}^n_+} u_h(\hat{x}_h)$$
   tax
   $$\text{s.t. } p^* \cdot \hat{x}_h \leq p \cdot e_h - t_h,$$

and 3. markets clear, i.e. $\sum_{h \in H} x_h^* = \sum_{h \in H} e_h$.

Theorem 4.8 (Second Welfare Theorem)

Consider a pure exchange economy (u, e) such that each u_h is continuous, increasing, and strictly quasi-concave, and endowments satisfy $\sum_{h \in H} e_{hn} > 0$ for all n. If x^* is an efficient allocation, then there exists transfers $t^* \in \mathbb{R}^H$ and prices $p^* \in \mathbb{R}^n_+$ s.t. (x^*, p^*, t^*) is an equilibrium with lump-sum transfers.

same as existence

<u>Proof</u> Consider the economy (u, x^*), i.e. with endowments equal to x^*. By the existence theorem, there exists p^* such that (x', p^*) is an equilibrium. All households can afford x^*_h, their endowment. So $u_h(x'_h) \geq u_h(x^*_h)$. Since x^* is efficient, we can't have any household with $u_h(x'_h) > u_h(x^*_h)$. We conclude $u_h(x'_h) = u_h(x^*_h)$. So (x^*, p^*) is an equilibrium.

Recall the budget constraints:

$$p^* \cdot x_h = p^* \cdot x^*_h \quad \leftarrow \text{step 1 (Robin Hood)}$$

$$p^* \cdot x_h = p^* \cdot e_h - t^*_h \quad \leftarrow \text{with taxes instead}$$

Let $t^*_h = -p^* \cdot x^*_h + p^* \cdot e_h$.

Check $\sum_h t^*_h = 0$:

$$\sum_h t^*_h = \sum_h \left[-p^* \cdot x^*_h + p^* \cdot e_h\right]$$

$$= p^* \cdot \underbrace{\sum_h \left[e_h - x^*_h\right]}_{=0 \text{ since } x^* \text{ is feasible}}$$

$$= 0.$$

So, (x^*, p^*, t^*) is an equilibrium with lump-sum taxes. $\square$

4.5 First welfare theorem (again)

Theorem 4.3 Consider a pure-exchange economy (u, e) with increasing utility functions u_h. ~~and~~ If (x^*, p^*) is an equilibrium, then x^* is an efficient allocation.

Proof Suppose $\hat{x}$ Pareto dominates x^*. Our goal is to prove $\hat{x}$ is infeasible.

Since $\hat{x}$ Pareto dominates x^*, we know $u_h(\hat{x}_h) \geq u_h(x^*_h)$, for all h, and $u_h(\hat{x}_h) > u_h(x^*_h)$ for at least one h. So $p^* \cdot \hat{x}_h \geq p^* \cdot x^*_h$ for all h and $p^* \cdot \hat{x}_h > p^* \cdot x^*_h$ for at least one h (since u_h is strictly increasing).

Summing up over households,

$$p^* \cdot \sum_h \hat{x}_h > p^* \cdot \sum_h x^*_h = p^* \cdot \sum_h e_h.$$

So $\hat{x}$ involves consuming more than e, i.e. $\hat{x}$ is infeasible. $\square$

Q33

i) Households

* two households, $h \in \{y, o\}$.
* m lab material endowment, price p_m
* k_h human capital, price r_y, r_o
* π_h consultancy h profits
* π_c cosmetics profits
* c_h cosmetic consumption, price p_c
* utility: $u(c_h)$

$$\max_{c_h} u(c_h)$$

$$\text{s.t. } p_c c_h = p_m m + r_h k_h \boxed{+ \pi_h + \frac{\pi_c}{2}}.$$

dividends

Consultancies

* M_h lab materials used
* K_h human capital used
* $s_h = f(M_h, K_h)$ output (consulting), price p_s.

Profit function:

$$\pi_h(p_s; p_m, r_h) = \max_{M_h, K_h} p_s f(M_h, K_h) - p_m M_h - r_h K_h$$

Cosmetics firm

* S buy consulting services
* $C = g(S)$ output

$$\pi_c(p_c; p_s) = \max_S p_c g(S) - p_s S$$

<u>Equilibrium</u> Prices $(p_m, r_y, r_o, p_s, p_c)$ and quantities $(c_h, M_h, K_h, S_h, S, C)$ constitute an equilibrium if the quantities solve the corresponding optimisation problem above, and markets clear:

c_h = c_y, c_o

S_h: S_y, S_o consulting services sold by young firm

S: consulting services used by cosmetics firm

$$2m = M_y + M_o \quad (p_m)$$
$$k_y = K_y \quad (r_y)$$
$$k_o = K_o \quad (r_o)$$
$$S_y + S_o = S \quad (p_s)$$
$$C = c_y + c_o \quad (p_c)$$

(ii) By Walras law, if markets do not clear, then (at least) one market has excess supply. Since only the labour markets do not clear, there must be excess ~~demand~~ supply in one of the labour markets.

(iii) ~~Ther~~ By the envelope theorem,

(A) $$\frac{\partial \pi_o(p_s; p_m, r_o)}{\partial p_m} = -M_o(p_s; p_m, r_o).$$

Now, π_0 is the upper envelope of linear (and ~~so~~ hence convex) functions, one per choice of (M_0, K_0),

$$(p_s; p_m, r_0) \longmapsto p_s f(M_0, K_0) - p_m M_0 - r_0 K_0$$

$$= (f(M_0, K_0), -M_0, -K_0) \cdot (p_s, p_m, r_0)$$

So π is a convex function.

So the left side of (A) is ~~dec~~ increasing in p_m.

$\Rightarrow$ the right side of (A) is increasing in p_m. So $M_0(p_s; p_m, r_0)$ is decreasing in p_m.

(iv) $V_0(K_0, p_s, p_m) = \max_{M_0} p_s f(M_0, K_0) - p_m M_0$

ambiguous $\pi_0(p_s; p_m, r_0) = \max_{K_0} V_0(K_0, p_s, p_m) - r_0 K_0$

I meant: $\frac{\partial f}{\partial x_1}(\hat{x})\big|_{\hat{x}=tx}$

(v) Hint: $\boxed{t\frac{\partial f}{\partial x_1}(tx)} = t\frac{\partial f}{\partial x_1}(x)$, for all t.

Recall: the question assumed $k_0 = 2k_y$.

By market clearing, $K_0 = 2K_y = 2k_y$.

F.O.C. M_h: $p_s \frac{\partial f(M_h, K_h)}{\partial M} - p_m = 0$

$$\Longleftrightarrow \frac{\partial f(M_h, K_h)}{\partial M} = \frac{p_m}{p_s}, \quad \text{for all } h \in \{y, o\}$$

Since RHS is the same for both firms, the LHS must be the same, i.e.

$$\frac{\partial f(M_y, K_y)}{\partial M} = \frac{\partial f(M_0, K_0)}{\partial M}$$

Since $K_0 = 2K_y$, and the hint, we get

$$\frac{\partial f(M_0, K_0)}{\partial M} = \frac{\partial f(M_0, 2K_y)}{\partial M} \quad \text{(since } K_0 = 2K_y\text{)}$$

$$= \frac{\partial f(\tfrac{1}{2}M_0, K_y)}{\partial M} \quad \text{(hint)}$$

$$= \frac{\partial f(M_y, K_y)}{\partial M} \quad \text{(} \cancel{\text{from}} \text{ FOC above)}$$

So, by decreasing marginal productivity,

$$\tfrac{1}{2}M_0 = M_y.$$

(vi) This is impossible.

Suppose there were such a tax scheme, with equilibrium perfume ~~quantity~~ supply $\hat{C}$.

Let C^* be the untaxed equilibrium perfume supply, and c_y^* and c_0^* the consumption quantities.

We want to rule out $\hat{C} > C^*$. If this were the case, we could allocate $\hat{c}_y = c_y^* + \frac{1}{2}(\hat{C} - C^*)$
and $\hat{c}_0 = c_0^* + \frac{1}{2}(\hat{C} - C^*)$
is feasible.

Since households only care about perfume,
$u_y(\hat{c}_y) > u_y(c_y^*)$ and $u_0(\hat{c}_0) > u_0(c_0^*)$.

So $(\hat{c}_y, \hat{c}_0)$ along with the other tax quantities (lab materials, etc.)

Pareto dominates (c_y^*, c_o^*).
So (c_y^*, c_o^*) is inefficient
– contradicts the first welfare
theorem. So the premise that
$\hat{c} > c^*$ is false.