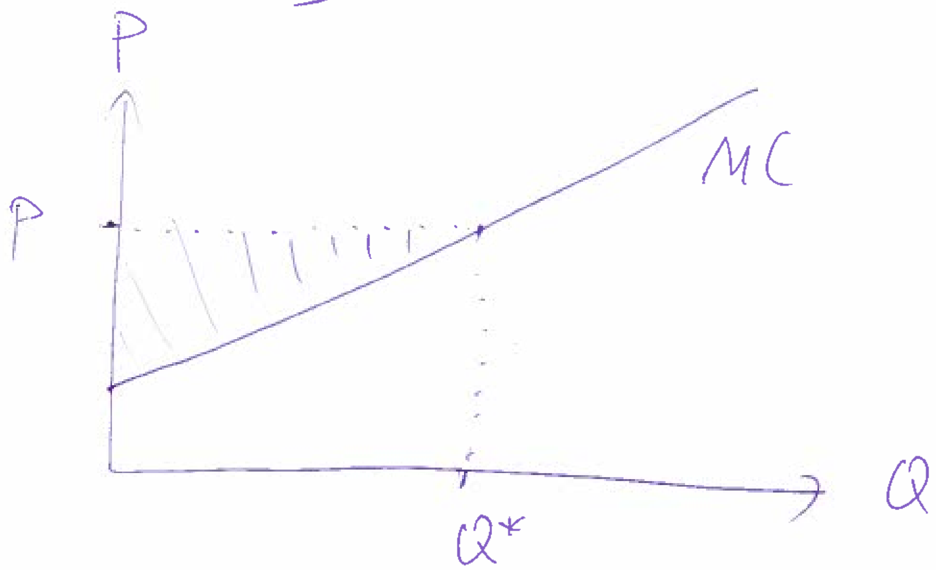
Theorem If $y(p,w)$ is the optimal supply policy in the firm's problem, then ~~for~~ all (p,w),

$$p = \left.\frac{\partial c(y,w)}{\partial y}\right|_{y=y(p,w)}.$$

Recall Econ 1



Proof Recall the Bellman equation

$$\pi(p;w) = \max_{y \in \mathbb{R}_+} py - c(y;w).$$

FOC: the answer. $\square$

We can also apply the envelope theorem to the Bellman equation:

$$\frac{\partial \pi(p;w)}{\partial p} = \left[\frac{\partial}{\partial p}\{py - c(y;w)\}\right]_{y=y(p;w)}$$

$$= [y]_{y=y(p;w)}$$

$$= y(p;w).$$ Same as before

and

$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\{py - c(y;w)\}\right]_{y=y(p;w)}$$

$$= \left[-\frac{\partial c(y;w)}{\partial w_i}\right]_{y=y(p;w)}.$$

2.5 Upper Envelopes with Constraints

Previously we studied value functions of the form

$$V(a) = \max_b v(a,b).$$

Now, we have a more complicated type of value function:

$$V(a) = \max_b v(a,b)$$
$$\text{s.t. } w(a,b) \geq 0.$$

Lagrangian: $\mathcal{L}(a,b,\lambda) = v(a,b) + \lambda w(a,b)$.

FOC b: $\dfrac{\partial \mathcal{L}(a,b,\lambda)}{\partial b} = \dfrac{\partial v(a,b)}{\partial b} + \lambda \left.\dfrac{\partial w(a,b)}{\partial b}\right| = 0$

at optimal (b, λ)

Theorem ("Constrained Envelope Theorem")

If V, v, w, b, λ are all differentiable functions and the constraint binds at a, i.e. $w(a, b(a)) = 0$, then

$$V'(a) = \left[\frac{\partial v(a,b)}{\partial a} + \lambda \frac{\partial w(a,b)}{\partial a}\right]_{\substack{b=b(a)\\ \lambda=\lambda(a)}}$$

<u>Proof</u> We eliminate the max by substituting in the policy:

$$\begin{aligned} V(a) &= v(a, b(a)), \\ &= v(a, b(a)) + \lambda(a) \underbrace{w(a, b(a))}_{=0} \\ &= L(a, b(a), \lambda(a)). \end{aligned}$$

Differentiating with the chain rule gives

$$\begin{aligned} V'(a) = \Bigg[& \frac{\partial L(a,b,\lambda)}{\partial a} + \overbrace{\frac{\partial L(a,b,\lambda)}{\partial b}}^{=0 \text{ by FOC}} b'(a) \\ & + \underbrace{\frac{\partial L(a,b,\lambda)}{\partial \lambda}}_{= w(a, b(a)) = 0} \lambda'(a) \Bigg]_{b=b(a),\ \lambda=\lambda(a)}. \end{aligned}$$

$$= \left.\frac{\partial L(a,b,\lambda)}{\partial a}\right|_{b=b(a),\ \lambda=\lambda(a)}$$

$$= \left[\frac{\partial v(a,b)}{\partial a} + \lambda \frac{\partial w(a,b)}{\partial a}\right]_{b=b(a),\ \lambda=\lambda(a)}.$$

$\square$

Recall the cost function

$$c(y;w) = \min_{x \in \mathbb{R}_+^{N-1}} w \cdot x \quad \text{s.t. } f(x) \geq y,$$

with Lagrangian

$$\mathcal{L}(y, w; x; \lambda) = w \cdot x - \lambda[f(x) - y].$$

Envelope theorem says:

$$\frac{\partial c(y;w)}{\partial y} = \left[\frac{\partial}{\partial y}\{w \cdot x - \lambda[f(x) - y]\}\right]_{\substack{\text{optimal} \\ (x,\lambda)}}$$

$$= \lambda(y;w). \leftarrow \text{The big idea behind Lagrange multipliers!}$$

$$\frac{\partial c(y;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\{w \cdot x - \lambda[f(x) - y]\}\right]_{\substack{\text{optimal} \\ (x,\lambda)}}$$

$$= x_i(y;w).$$

<u>Theorem 2.6</u> If v is convex and w is quasi-concave, then

$$V(a) = \min_b v(a,b) \text{ s.t. } w(a,b) \geq 0$$

is a convex function. See notes for a concave version.

Proof

We would like to prove that for all t, a, a',

$$\underbrace{tV(a) + (1-t)V(a')}_{\text{line}} \geq \underbrace{V(ta + (1-t)a')}_{\text{curve}}.$$

Start from left side:

$$tV(a) + (1-t)V(a')$$

$$= t\,v(\underbrace{a, b(a)}_{x}) + (1-t)\,v(\underbrace{a', b(a')}_{x'})$$

$$\geq v(tx + (1-t)x') \quad \text{since } v \text{ is convex}$$

$$= v(ta + (1-t)a', \; t\,b(a) + (1-t)\,b(a'))$$

$$\geq v(ta + (1-t)a', \; b(ta + (1-t)a'))$$

$$= V(ta + (1-t)a').$$

respects the constraint since w is quasi-concave

$\square$

Notes: recall the constraints are

$$w(a, b) \geq 0.$$

So $w(a, b(a)) \geq 0$ and $w(a', b(a')) \geq 0$.

Q-C $\Rightarrow w(ta + (1-t)a', \; t\,b(a) + (1-t)\,b(a')) \geq 0$

__Theorem 2.7__ If the production function f is concave, then the cost function is convex in the output target, i.e. $c(\cdot, w)$ is convex for all w.

__Proof__ Problem: $g(w, x) = w \cdot x$ is not a convex function!

Solution: hold prices w fixed.

Recall

trick

$$c(y; \cancel{x}) = \min_{x \in \mathbb{R}^{N-1}_{+}} w \cdot x$$

$$\text{s.t.} \quad f(x) \geq y.$$

Since f is concave and $-y$ is concave (linear functions are concave), the sum $f(x) - y$ is a concave function of (x, y).

So the constraint is quasi-concave.

The objective $v(y; x) = w \cdot x$ is linear, and hence convex.

So the theorem establishes that $c(y; w)$ is ~~concave~~ convex in y for all w. □

Chapter 3 - Consumption

3.1 Utility functions

Survey:

(i) 3 bed in Leith vs 2 bed in New Town

⋮

lasts forever.

Possible choices: $x \in \mathbb{R}^N_+$.

<u>Def</u> Consider two choices $x, y \in \mathbb{R}^N_+$.
If the consumer (in the infinite survey) says the like x better than y (weakly), we write $\boxed{x \succsim y.}$ all we need
If it is strict preference, we write $x \succ y$. Note: strict preference means $x \succsim y$ but $y \not\succsim x$. If $x \succsim y$ and $y \succsim x$, we write $x \sim y$ ("indifferent").

<u>Def</u> A <u>utility function</u> is a function $u : \mathbb{R}^N_+ \to \mathbb{R}$. We say u <u>represents</u> the preferences $\succsim$ if for all $x, y \in \mathbb{R}^N_+$

$$u(x) \geq u(y) \iff x \succsim y.$$

Possible assumptions about $\succsim$:

* <u>complete</u>: for all $x, y \in \mathbb{R}^N_+$, either $x \succsim y$ or $y \succsim x$. (or both)
* <u>reflexive</u>: for all $x \in \mathbb{R}^N_+$, $x \succsim x$.
* <u>transitive</u>: for all $x, y, z \in \mathbb{R}^N_+$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.
* <u>continuous</u>: all upper and lower contour sets are closed sets. (An upper contour set is $U(x) = \{y \in \mathbb{R}^N_+ : y \succsim x\}$.)

<u>Theorem 3.1</u> Consider a preference relation $\succsim$. There exists a continuous utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$ that represents $\succsim$ if and only if $\succsim$ is complete, reflexive, transitive, and continuous.

Another possible assumption:

* <u>convexity</u>: all upper contour sets are convex sets.

$\Leftrightarrow$ corresponding utility function is quasi-concave.

Note: no good reason to think u is concave.

3.2 Time Preference

Example 3.1 Suppose there is one good ($N=1$) "methamphetamine" and you can consume 0 or 1 each day over four days ($T=4$ time periods). $2^4 = 16$ options including:

only changes (on 2nd day)

before $\begin{cases} x = 0000 \\ y = 0011 \end{cases}$

after $\begin{cases} x' = 0100 \\ y' = 0111 \end{cases}$ ← cold turkey (bad)

$x \succsim y$ and $y' \succsim x'$, what most (sensible) people prefer.

These pref's are <u>not time-separable</u>.

The day that changes (day 2) has the same choices for x and y and also x' and y'. "From to the same thing, to the same thing."

Time-separable says: such a change should not reverse preferences.

(*) <u>Theorem 3.3</u> if $\succsim$ are preferences over $\mathbb{R}_+^{NT}$ are complete, reflexive, transitive, time-separable, strictly increasing, $T \geq 3$ then there exists continuous utility functions

$$u_1, \ldots, u_T$$

such that $\succsim$ are represented by

day 1 consumptions

$$U(x) = \underbrace{u_1(x_1) + u_2(x_2) + \cdots + u_T(x_T)}_{\text{additively separable}}.$$

day 2

<u>Proof</u> See Debreu (1960). $\square$

Discounted utility:

$$U(x) = u(x_1) + \beta u(x_2) + \beta^2 u(x_3) + \cdots + \beta^{T-1} u(x_T)$$

Cake-eating problem:

$$V(k) = \max_{x_1, \ldots, x_T \geq 0} u_1(x_1) + \cdots + u_T(x_T)$$

size of cake

$$\text{s.t. } x_1 + \cdots + x_T = k.$$

Or at date t:

$$V_t(k_t) = \max_{x_t, \ldots, x_T \geq 0} u_t(x_t) + \cdots + u_T(x_T)$$

$$\text{s.t. } x_t + \cdots + x_T = k_t.$$

Bellman equation:

$$V_t(k_t) = \begin{cases} u_T(k_T) & \text{if } t = T, \\ \max\limits_{x_t,\, k_{t+1}} u(x_t) + V_{t+1}(k_{t+1}) & \text{if } t < T \\ \text{s.t. } x_t + k_{t+1} = k_t. \end{cases}$$

HW: read the maths behind time-sep (para 1-3)

<u>AND</u> principle of optimality

3.3 Utility Maximisation

$$\underbrace{v(p, m)}_{\text{indirect utility function}} = \max_{x \in \mathbb{R}^N_+} u(x)$$

$$\text{s.t. } p \cdot x \leq m$$

money

More philosophically sound version:

$$v^*(p, e) = \max_{x \in \mathbb{R}^N_+} u(x)$$

$$\text{s.t. } \underbrace{p \cdot x}_{\text{buy consumption}} \leq \underbrace{p \cdot e}_{\text{sell endowments}}$$

endowments

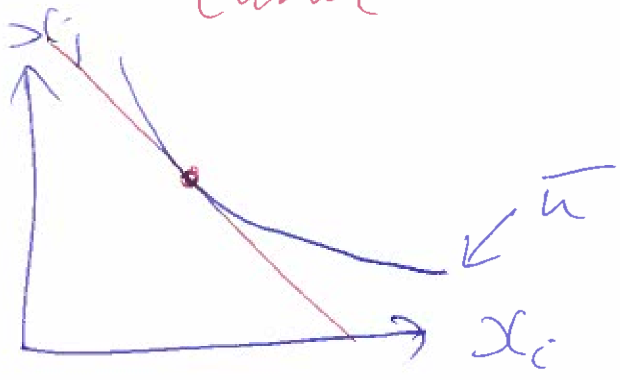
FOC x_i: $\left[\frac{\partial u(x)}{\partial x_i} - \lambda p_i\right]_{\substack{\text{optimal} \\ (x,\lambda)}} = 0$

Lagrange multiplier on budget constraint.

rearrange: $\left.\frac{\partial u(x)}{\partial x_i}\right|_{\text{optimal } x} = \lambda \Big| p_i$ optimal λ

$$\frac{\text{rearranged } x_i}{\text{rearranged } x_j} \qquad \frac{\frac{\partial u(x)}{\partial x_i}}{\frac{\partial u(x)}{\partial x_j}} = \frac{p_i}{p_j}.$$

by implicit function theorem ← (-ve) slope of indifference curve (-ve) slope of budget constraint



Detour F.3:

On an indifference curve $x_j(x_i)$,

$$u(\underbrace{x_{-ij}}_{\text{holding all other quantities fixed}},\ x_i,\ x_j(x_i)) = \bar{u}$$

for all x_i.

Idea: two functions are equal
$\Rightarrow$ same derivatives (0!)

Derivative of LHS:

$$\left[\frac{\partial u(x_{-ij}, x_i, x_j)}{\partial x_i} + \frac{\partial u(x_{-ij}, x_i, x_j)}{\partial x_j} \frac{d x_j(x_i)}{d x_i} \right] \Bigg|_{x_j = x_j(x_i)} = 0$$

Rearrange: $$\frac{d x_j(x_i)}{d x_i} = - \frac{\frac{\partial u(x_{-ij}, x_i, x_j)}{\partial x_i}}{\frac{\partial u(x_{-ij}, x_i, x_j)}{\partial x_j}}$$

3.4 Consumer's value & Policy Functions

Recall $$v(p,m) = \max_{x \in \mathbb{R}^N_+} u(x) \quad \text{s.t. } p \cdot x \leq m$$

Envelope theorem:

$$\frac{\partial v(p,m)}{\partial p_i} = - \lambda(p,m) x_i(p,m)$$

Detail: $$\mathcal{L}(p,m,x,\lambda) = u(x) - \lambda[p \cdot x - m]$$

$$\frac{\partial v(p,m)}{\partial p_i} = \left[\frac{\partial}{\partial p_i} \mathcal{L}(p,m,x,\lambda)\right]_{\text{optimal } (x,\lambda)}$$

$$= \left[\frac{\partial}{\partial p_i} \{ u(x) - \lambda p \cdot x + \lambda m \} \right]_{\substack{\text{optimal} \\ (x,\lambda)}}$$

$$= [-\lambda x_i]_{\text{optimal } (x,\lambda)}$$

$$= -\lambda(p,m)\, x_i(p,m).$$

$$\frac{\partial v(p,m)}{\partial m} = \lambda(p,m).$$

So $\cancel{\frac{\partial v(p,m)}{\partial p_i} =}$ $\underbrace{x_i(p,m)}_{\text{optimal choice}} = \underbrace{-\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}}_{\text{marginal values}}$

— yuck!

Terminology:

* normal good: A good x_i is normal at (p,m) if demand increases when wealth increases, i.e. $x_i(p,m)$ is increasing in m.
* inferior good: $x_i(p,m)$ is decreasing in m at (p,m).
* Giffen good: $x_i(p,m)$ is increasing in p_i.
* substitutes: x_i and x_j are substitutes if $x_i(p,m)$ is increasing in p_j.

* <u>complements</u>: x_i and x_j are complements if $x_i(p, m)$ is decreasing in p_j

3.5 Expenditure Function & Policy Functions

Plan: shut down income effect.

Expenditure function:

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}^N_+} p \cdot x \quad \text{s.t. } u(x) \geq \bar{u}$$

utility target

$$= p \cdot h(p, \bar{u})$$

policy function ("Hicksian demand")

Bellman equation:

$$v(p, m) = \max_u u \quad \text{s.t. } e(p, u) = m.$$

We apply the envelope theorem to the expenditure minimisation problem:

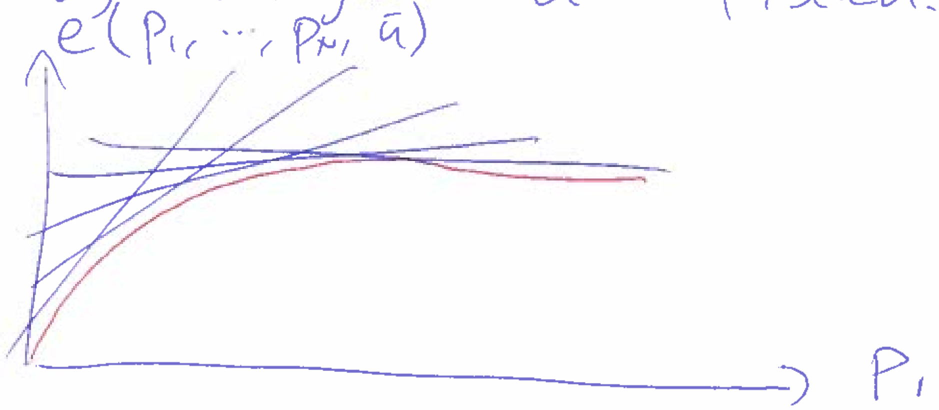
$$\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u)$$

no Lagrange multiplier (YAY!)

$$\frac{\partial e(p, u)}{\partial u} = \mu(p, u)$$

Lagrange multiplier for $u(x) \geq \bar{u}$

Now e is the lower envelope of linear function (one function for each x) when holding the utility target $\bar{u}$ fixed.



So $e(p, \bar{u})$ is concave in prices

Combining, we deduce

$$\frac{\partial h_i(p, u)}{\partial p_i} = \frac{\partial^2 e(p, u)}{\partial p_i^2} < 0.$$

envelope

concavity