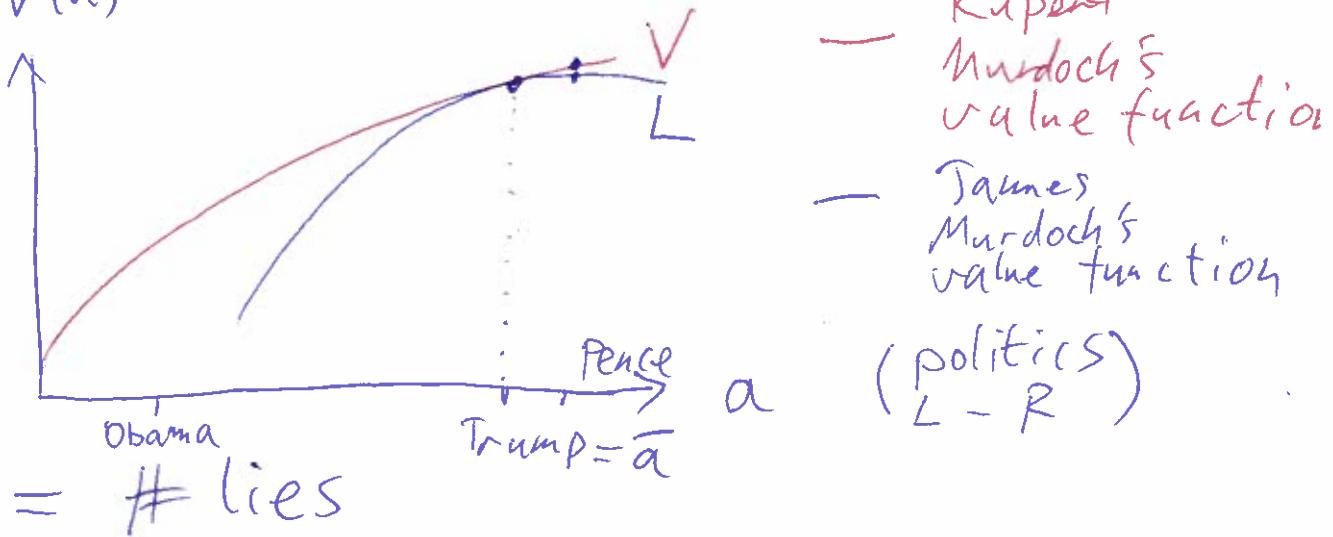


Proof $v(a) \leftarrow \text{Profit}$



$b(\bar{a}) \leftarrow \text{Rupert Murdoch's last words}$

$$L(a) = v(a, b(\bar{a}))$$

Notice:

- ① $V(\bar{a}) = L(\bar{a})$
- ② $V(\bar{a}) \geq L(a)$ for all a .

Therefore, \bar{a} minimises $V(a) - L(a)$,
so the F.O.C.

$$V'(\bar{a}) = L'(\bar{a})$$

holds.

$$\text{Now, } L'(\bar{a}) = \frac{\partial}{\partial a} v(a, b(\bar{a}))$$

$$= \frac{\partial}{\partial a} v(a, b) \Big|_{\begin{array}{l} a=\bar{a} \\ b=b(\bar{a}) \end{array}}$$

We conclude $V'(\bar{a}) = \frac{\partial}{\partial a} v(a, b) \Big|_{b=b(\bar{a})}$. \square

Proof (chain rule):

Recall $v(a) = v(a, b(a))$.

$$v'(a) = \frac{\partial v(a, b)}{\partial a} \Big|_{b=b(a)} + \underbrace{\frac{\partial v(a, b)}{\partial b} \Big|_{b=b(a)} b'(a)}_{\text{matrix multiplication}}$$

\curvearrowleft direct effect \curvearrowleft indirect effect

High school: if $h(x) = f(g(x))$,
then $h'(x) = f'(g(x))g'(x)$.

$$v'(a) = \begin{bmatrix} \frac{\partial v(a, b)}{\partial a} & \frac{\partial v(a, b)}{\partial b} \end{bmatrix} \begin{bmatrix} I \\ b'(a) \end{bmatrix}.$$

Recall that $b(a)$ solves

$\max_b v(a, b)$,

so it satisfies the F.O.C. $\frac{\partial v(a, b)}{\partial b} \Big|_{b=b(a)} = 0$.

So we conclude

$$v'(a) = \frac{\partial v(a, b)}{\partial a} \Big|_{b=b(a)} \quad \square$$

E.g. Consider a manager choosing #workers l to hire in response to wages w . Profit function:

$$\pi(w) = \max_l 10\sqrt{l} - wl.$$

Q: What is $\pi'(w)$?

A: with envelope theorem

$$\begin{aligned}\pi'(w) &= \left[\frac{\partial}{\partial w} (10\sqrt{l} - wl) \right]_{l=l(w)} \\ &= [-l]_{l=l(w)} \\ &= -l(w).\end{aligned}$$

A: without envelope theorem

① Get rid of max. FOC for l :

$$10 \cdot \frac{1}{\sqrt{l}} \cdot \frac{1}{2} - w = 0$$

$$\Leftrightarrow \frac{5}{\sqrt{l}} = w$$

$$\Leftrightarrow \frac{\sqrt{l}}{5} = \frac{1}{w}$$

$$\Leftrightarrow \sqrt{l} = \cancel{\frac{5}{w}}$$

$$\Leftrightarrow l(w) = \frac{25}{w^2}.$$

② Substitute into the manager's problem:

$$\begin{aligned}\pi(w) &= 10\sqrt{l(w)} - wl(w) \\ &= 10\sqrt{\frac{25}{w^2}} - w \cdot \frac{25}{w^2}\end{aligned}$$

$$= \frac{50}{w} - \frac{25}{w^2}$$

$$= \frac{25}{w}.$$

② Calculus.

$$\pi'(w) = -\frac{25}{w^2}.$$

③ Inspiration: $\frac{25}{w^2} = l(w)$.

$$\text{So } \pi'(w) = -l(w).$$

More abstractly, \leftarrow envelope theorem

$$\frac{\partial \pi(p, w)}{\partial p} = \left[\frac{\partial}{\partial p} \{ pf(x) - w \cdot x \} \right]_{x=x(p,w)}$$

$$= [f(x)]_{x=x(p,w)}$$

$$= f(x(p,w))$$

$$= y(p) \leftarrow \begin{matrix} \text{optimal output} \\ \text{quantity} \end{matrix}$$

$$\frac{\partial \pi(p, w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i} \{ pf(x) - w \cdot x \} \right]_{x=x(p,w)}$$

$$= [-x_i(p)]_{x=x(p,w)}$$

$$= -x_i(p,w).$$

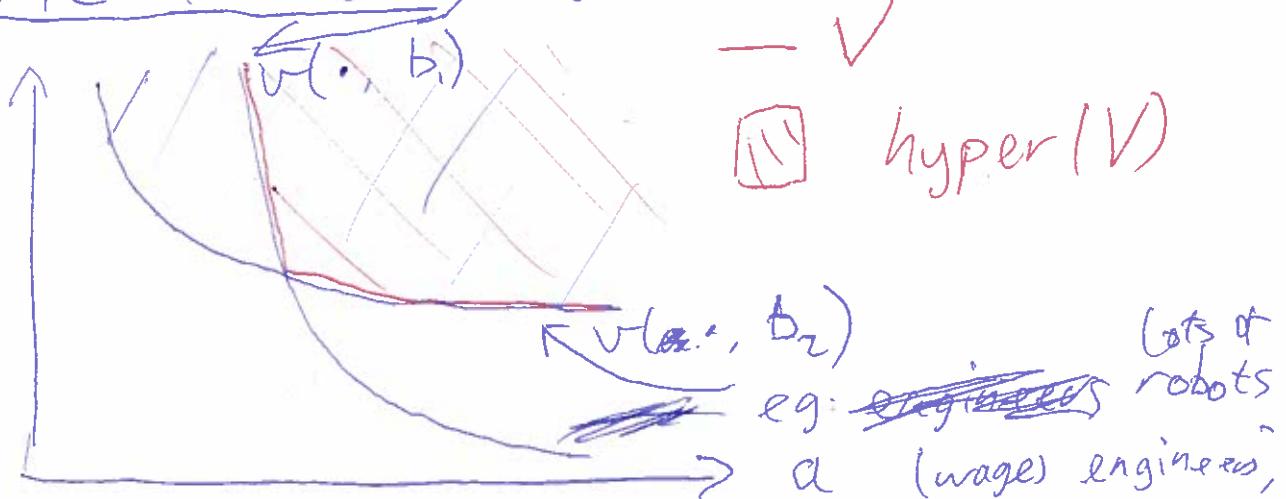
We can differentiate again:

$$\frac{\partial^2 \pi(p, w)}{\partial p^2} = \frac{\partial y^*(p,w)}{\partial p}$$

$$\frac{\partial \pi(p, w)}{\partial w_i^2} = \cancel{-} - \frac{\partial x_i(p, w)}{\partial w_i}.$$

Theorem 2.2 Suppose $V(a) = \max_b v(a, b)$ is the value function and each $v(\cdot, b)$ is a convex function (one function for each b). Then V is a convex function.

Geometric Proof eg: ~~robots~~ lots of engineers



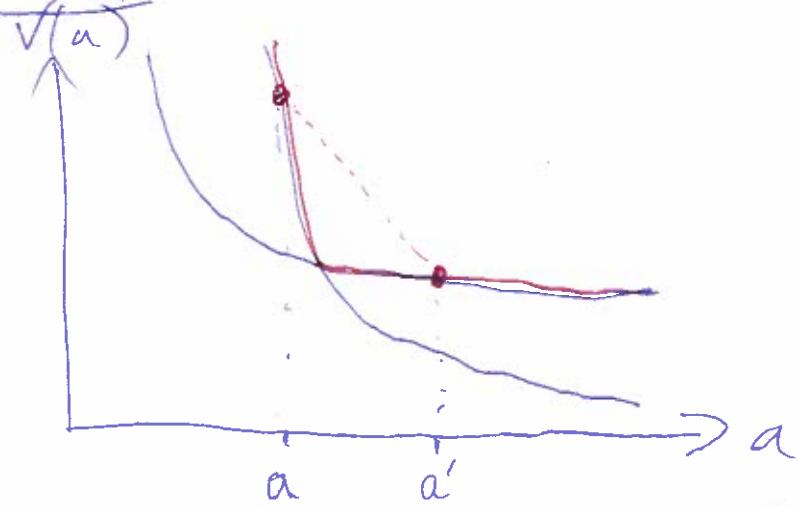
* By assumption, $\text{hyper}(v(\cdot, b_1))$ and $\text{hyper}(v(\cdot, b_2))$ are convex sets.

By Theorem D.1, $\text{hyper}(v(\cdot, b_1)) \cap \text{hyper}(v(\cdot, b_2))$ is a convex set, where we know

$$\text{hyper}(V) = \text{hyper}(v(\cdot, b_1)) \cap \text{hyper}(v(\cdot, b_2))$$

So V is a convex function. \square

Algebraic Proof



Recall Theorem D.6 says that V is a convex function if and only if

$$tV(a) + (1-t)V(a') \geq V(ta + (1-t)a')$$

for all $t \in [0, 1]$ and all a, a' .

Start from the left side:

$$tV(a) + (1-t)V(a')$$

$$= t\upsilon(a, b(a)) + (1-t)\upsilon(a', b(a'))$$

$$\geq t\upsilon(a, b(\underbrace{ta + (1-t)a'}_{\text{wrong choice!}})) + (1-t)\upsilon(a', b(ta + (1-t)a'))$$

$$\geq t\upsilon(a, b(ta + (1-t)a')) + (1-t)\upsilon(a', b(ta + (1-t)a')) \quad \begin{matrix} \text{same wrong choice} \\ \text{use convexity of objective} \end{matrix}$$

$$\geq \upsilon(ta + (1-t)a', b(ta + (1-t)a')) \quad \square$$

Theorem 2.3 For every production function f , the firm's profit function π is convex. Hence if π is smooth, then

$$\underbrace{\frac{\partial \pi(p; w)}{\partial p}}_{\text{slope of supply curve}} \geq 0 \quad \text{and} \quad \underbrace{\frac{\partial \pi(p; w)}{\partial w}}_{\text{slope of factor demand curve}} \leq 0.$$

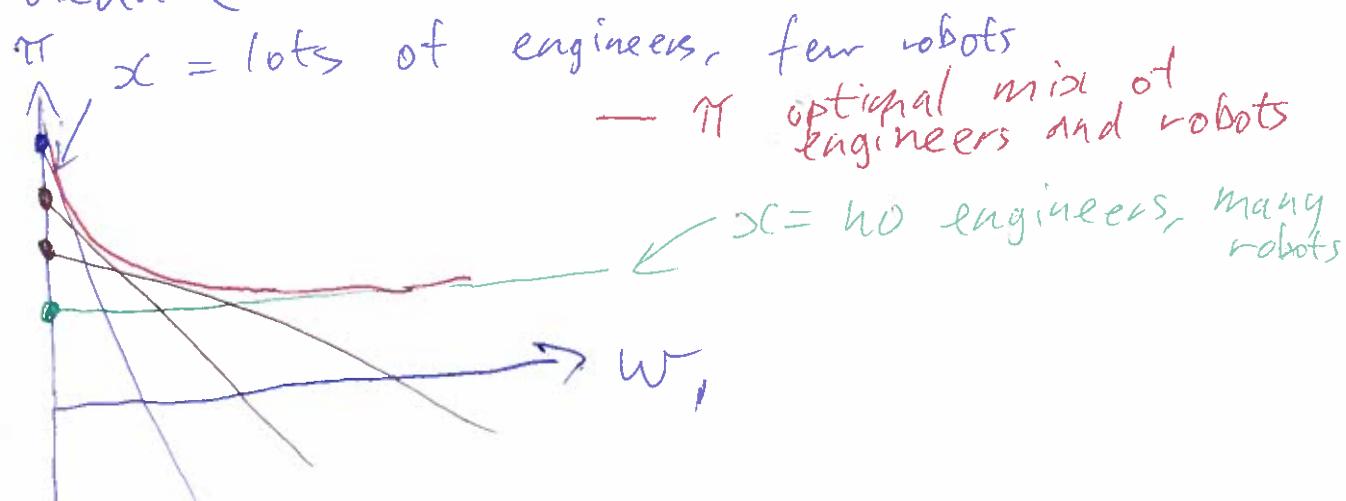
Proof Recall $\pi(p; w) = \max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$.

$$\text{Let } v(p, w; x) = \underbrace{pf(x)}_a - \underbrace{w \cdot x}_b. \Rightarrow \pi(p; w) = \max_x v(p, w; x)$$

We would like to use the previous theorem to prove that π convex. We need to check each $v(\cdot; x)$ is a convex function. In fact, ~~v(p, w; x)~~ $v(p, w; x)$ is linear in (p, w) . Rewrite

$$v(p, w; x) = (p, w) \cdot (f(x), -x).$$

We deduce that π is a convex function.

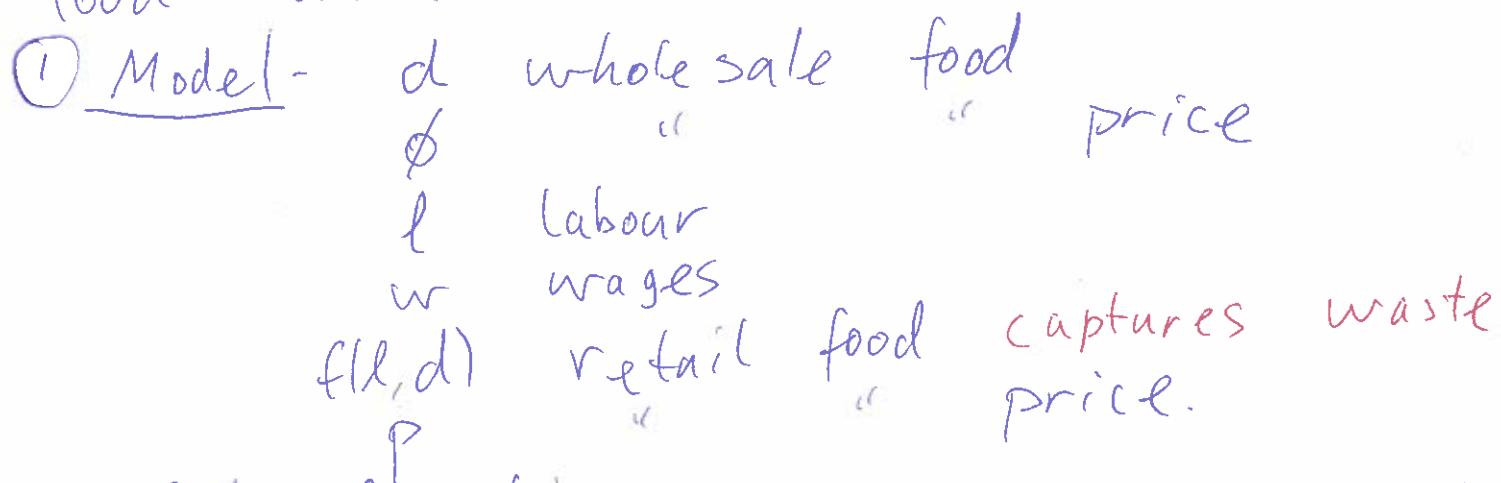


We previously used the envelope theorem to prove

$$\frac{\partial \pi(p; w)}{\partial p} = y(p; w) \text{ and } \frac{\partial \pi(p; w)}{\partial w_i} = -x_i(p, w).$$

By Theorem D.3, the left sides are increasing in p and w_i respectively.
So the right sides, $y(p; w)$ and $-x_i(p, w)$ are increasing in p and w_i respectively.
So $y(p; w)$ is increasing in p and
 $x_i(p; w)$ is decreasing in w_i . \square

Example 2.4 Consider a supermarket that buys wholesale & labour and sells retail food. More labour leads to less food waste.



Profit function:
 $\pi(p, \phi, w) = \max_{l, d} p f(l, d) - wl - \phi d.$

② π is convex. For each possible choice (l, d) , the firm's objective is linear in (p, ϕ, w) . Since linear functions

the upper envelope is convex.

③ If wholesale prices increase,
then wholesale demand decreases.

By the envelope theorem,

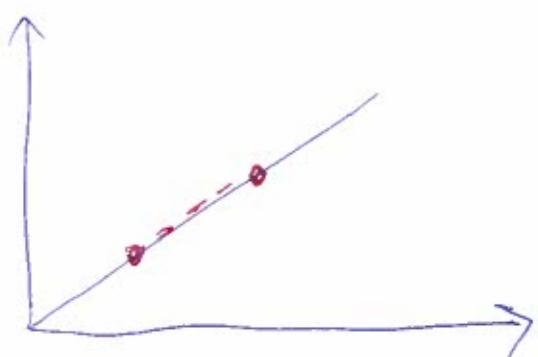
$$\frac{\partial \pi(p, \phi, w)}{\partial \phi} = \left[\frac{\partial}{\partial \phi} \{ p f(l, d) - w l - \phi d^2 \} \right]$$

$l = l(p, \phi, w)$
 $d = d(p, \phi, w)$

$$= [-d] \quad d = d(p, \phi, w)$$
$$= -d(p, \phi, w).$$

Since π is convex, the left side
is increasing in ϕ . So $d(p, \phi, w)$
is decreasing in ϕ .

Note: linear functions are convex
functions (and concave functions).



but not "strictly
convex", etc.

2.4 Cost functions & Dynamic Programming

virus: AA G G C A T G C
 $\underbrace{\quad}_{\quad} \underbrace{\quad}_{\quad}$

A A C G C A T A C

Eg: Profit functions & cost functions

$$\pi(p; w) = \max_{x \in \mathbb{R}_+^{N-1}} p f(x) - w \cdot x$$

Let $c(y; w) = \min_{x \in \mathbb{R}_+^{N-1}} w \cdot x$
s.t. $f(x) \geq y$

output \rightarrow
target

be the cost function. We can rewrite the profit function using the cost function:

$$\pi(p; w) = \max_y p y - c(y; w).$$

↙ a Bellman equation

In macro,

$$V(a) = \max_{c, c'} u(c) + \beta V(c')$$

s.t. $c + c' = a$.

Let $W(a) = u(a)$. Then

$$V(a) = \max_{c, a'} u(c) + \beta W(a') \text{ s.t. } c + a' = a.$$

↙ Bellman equations

Lemma (Principle of Optimality) — We didn't mess up the Bellman eq?

The two ~~two~~ definitions of π give the same function.

Proof We could more easily do guess & verify... but $\max_{x \in \mathbb{R}_+^{N-1}} p f(x) - w \cdot x$ these are cheating!

$$= \max_{y \in \mathbb{R}_+, x \in \mathbb{R}_+^{N-1}} p f(x) - w \cdot x \\ \text{s.t. } f(x) = y,$$

$$= \max_{y \in \mathbb{R}_+} \left[\max_{x \in \mathbb{R}_+^{N-1}} p f(x) - w \cdot x \right] \\ \text{s.t. } f(x) = y$$

$$= \max_{y \in \mathbb{R}_+} \left[\max_{x \in \mathbb{R}_+^{N-1}} p y - w \cdot x \right] \\ \text{s.t. } f(x) = y$$

$$= \max_{y \in \mathbb{R}_+} \left\{ p y + \left[\max_{x \in \mathbb{R}_+^{N-1}} -w \cdot x \right] \right\} \\ \text{s.t. } f(x) = y$$

$$= \max_{y \in \mathbb{R}_+} \left\{ p y + \left[\min_{x \in \mathbb{R}_+^{N-1}} w \cdot x \right] \right\} \\ \text{s.t. } f(x) = y$$

$$= \max_y p y - c(y; w).$$

□