2 Production

N goods

1 output, $y \in \mathbb{R}_+$ ← notation in Appendix B

N−1 inputs, $x \in \mathbb{R}^{N-1}_+$.

$f(x)$ production function

$$f : \underbrace{\mathbb{R}^{N-1}_+}_{\text{inputs}} \to \underbrace{\mathbb{R}_+}_{\text{outputs}}$$

$$y = f(x)$$

What assumptions should we make about f?

* <u>Shutdown</u>: $f(0) = 0$. $\quad 0 = (\underbrace{0, 0, \ldots, 0}_{N-1})$

eg: f is the VHS production function

* <u>Free disposal (monotonicity)</u>:
If $x \geq x'$ (i.e. $x_n \geq x'_n$ for all goods n) then $f(x) \geq f(x')$.

* <u>Smoothness</u>: f is twice continuously differentiable. Each partial derivative $\frac{\partial}{\partial x_i} f(x)$ is called the <u>marginal productivity</u> of x_i.

What assumption(s) would lead to (weakly) increase marginal cost?

* <u>Decreasing marginal productivity</u>

We say that a production function f has weakly decreasing marginal productivity in good 1 if $\frac{\partial f}{\partial x_1}(x)$ is weakly decreasing as x_1 increases (keeping $x_2, \ldots, x_{N-1}$ fixed).

* <u>weakly decreasing returns to scale</u>:

For all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$, cheating!

$$\underbrace{f(tx)}_{\text{1 big firm}} \leq \underbrace{t f(x)}_{t \text{ small firms}}.$$

* <u>constant returns to scale</u>:

For all $x \in \mathbb{R}_+^{N-1}$ and all $t > 0$,

$$f(tx) = t f(x).$$

* weakly increasing returns to scale

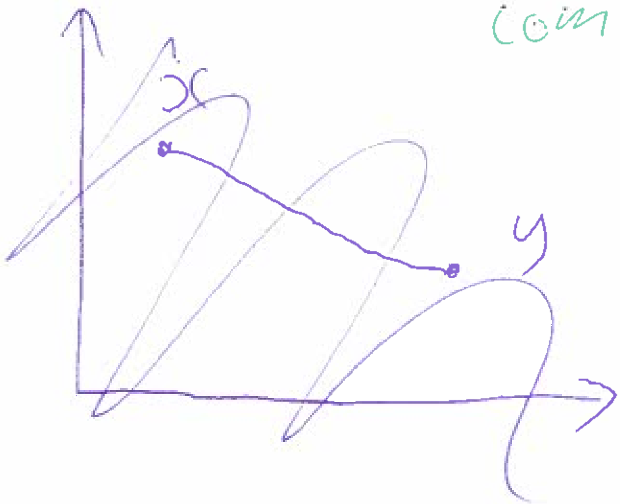
For all $x \in \mathbb{R}^{N-1}_{+}$ and all $t > 1$,

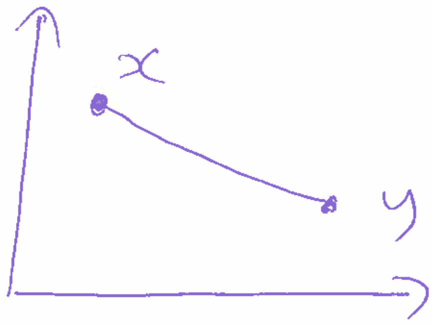
$$f(tx) \geq t f(x).$$

D Convex Geometry

Def A closed interval between two points $x, y \in \mathbb{R}^n$ is defined as

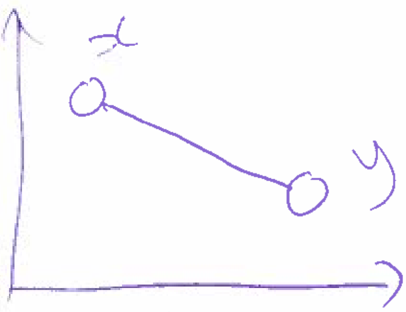
$$[x,y] = \{\underbrace{ax + (1-a)y}_{\text{convex combination}} : a \in \underbrace{[0,1]}_{\text{all numbers between 0 and 1 inclusive.}}\}$$





Open interval: $(x,y) = \{ax + (1-a)y : a \in (0,1)\}$

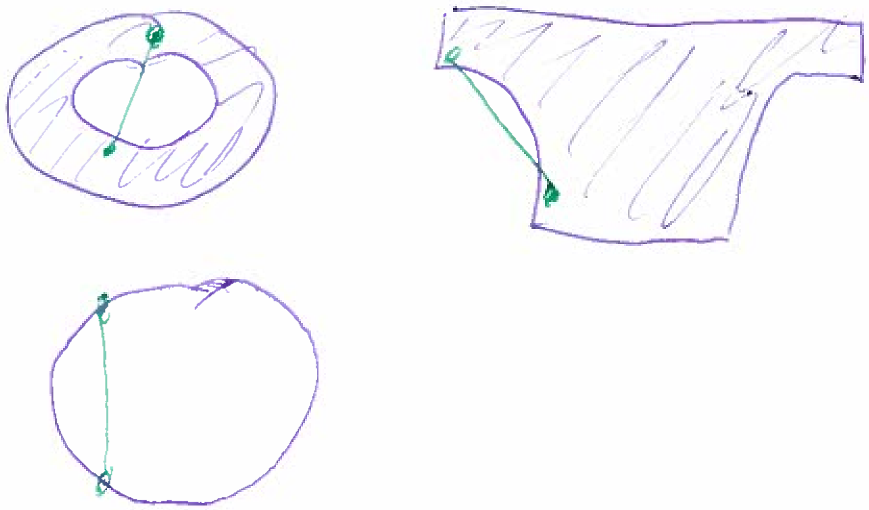
excludes the endpoints



$[x,y)$ and $(y,x]$ are similar.

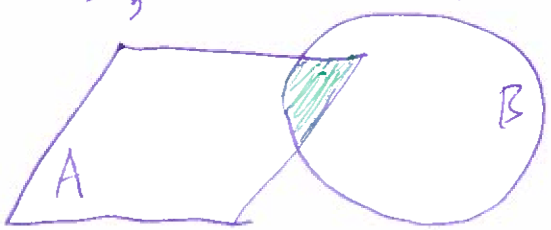
Def Let $X \subseteq \mathbb{R}^n$. We say X is a convex set if for all $x, y \in X$, then $[x, y] \subseteq X$.

e.g. NOT convex:



e.g. are convex:

Theorem If A and B are convex sets, then $A \cap B$ is a convex set.



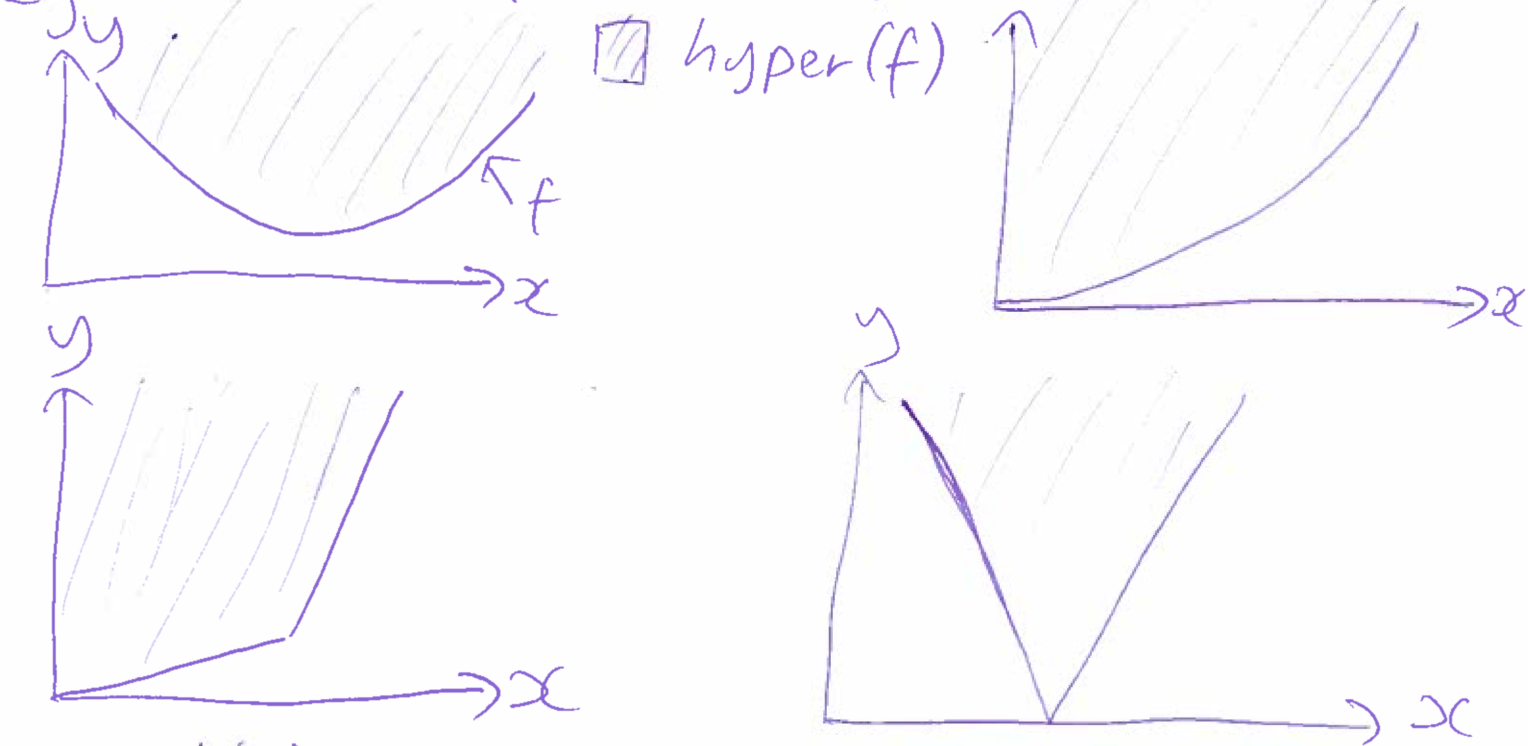
Proof: Pick any $x, y \in A \cap B$. Notice that $x, y \in A$. Since A is a convex set, $[x, y] \subseteq A$. Similarly $[x, y] \subseteq B$. Combining, we deduce that $[x, y] \subseteq A \cap B$. So $A \cap B$ is a convex set. $\square$

<u>Def</u> $f : X \to \mathbb{R}$ is a <u>convex function</u> if its <u>hypergraph</u>

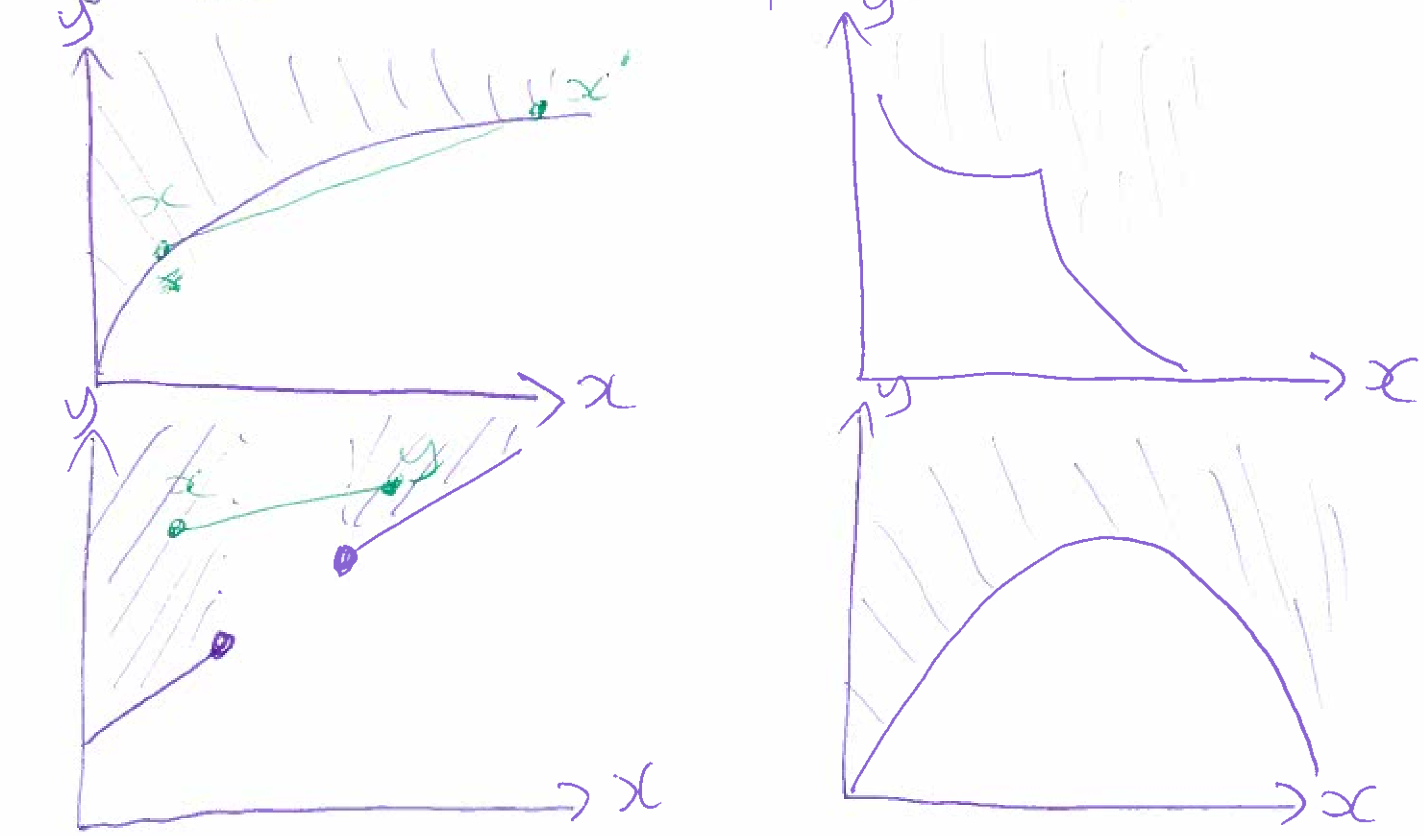
$$\{(x,y) : x \in X,\ y \geq f(x)\} = \mathrm{hyper}(f)$$

is a convex set.

eg: convex functions:



e.g. <u>NOT</u> convex functions:



Def $f: X \to \mathbb{R}$ is a **concave function** if its **hypograph**

$$\text{hypo}(f) = \{(x,y) : x \in X,\ y \leq f(x)\}$$

is a convex set.

eg:

is ~~convex~~ concave

Both concave & convex

Neither concave nor convex.

WARNING: There is no such thing as a concave set.

Theorem if $f: X \to \mathbb{R}$ is a convex function and X is open set then f is continuous.

theorem Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is a ~~a~~ convex function if and only if its derivative f' is weakly increasing.

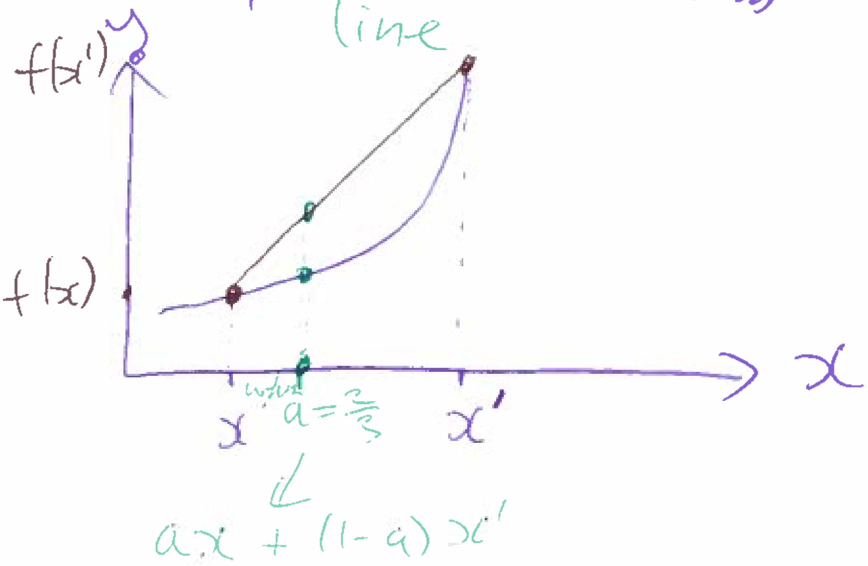
Theorem Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is a convex function if and only if $f''(x) \geq 0$ for all x.

concave

Theorem $f: X \to \mathbb{R}$ is a convex function if and only if for all $x, x' \in X$ and all $a \in (0,1)$,

$$a f(x) + (1-a) f(x') \geq f(ax + (1-a)x').$$

line $\qquad\qquad$ curve



"line is above curve"

Back to production functions:

Assume:

* f is a concave function.

$\Rightarrow$ weakly decreasing marginal productivity

and $\Rightarrow$ weakly decreasing or constant returns to scale.

Claim Suppose $f: \mathbb{R}_+^{N-1} \to \mathbb{R}_+$ is concave and $f(0) = 0$. Then f has weakly decreasing returns to scale.

Proof Specifically, we want to prove that $f(tx) \leq t f(x)$ for all $t > 1$ and all $x \in \mathbb{R}_+^{N-1}$.

Let $s = \frac{1}{t}$, which means $s \in (0,1)$.

By the theorem,

$$\cancel{s\, f(tx) \leq f(stx}$$

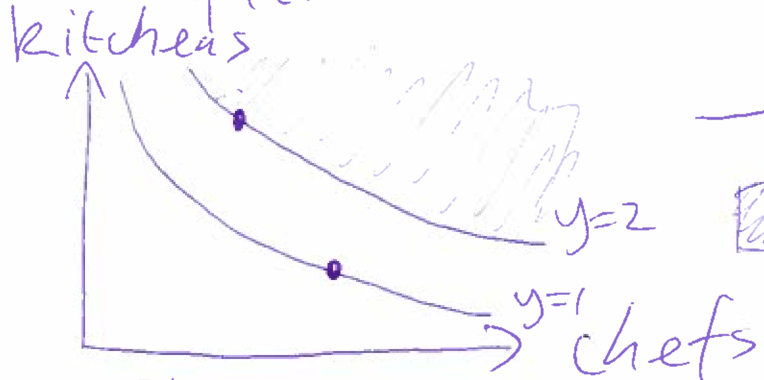
$$s f(tx) + (1-s) f(0) \leq f(stx + (1-s)0)$$

chose $a = s$ and $x = x$ and $x' = 0$

Simplifying,

$$\begin{aligned}
s f(tx) + 0 &\leq f(stx + 0) \\
\tfrac{1}{t} f(tx) &\leq f(\tfrac{1}{t} tx) \\
\tfrac{1}{t} f(tx) &\leq f(x) \\
f(tx) &\leq t f(x).
\end{aligned}$$

$\square$



— isoquants

an upper contour set

Def The upper contour set for output level y is

$$V(y) = \{x \in \mathbb{R}_+^{N-1} : f(x) \geq y\} = f^{-1}([y, \infty)).$$

lower $\quad \leq \quad (-\infty, y]$

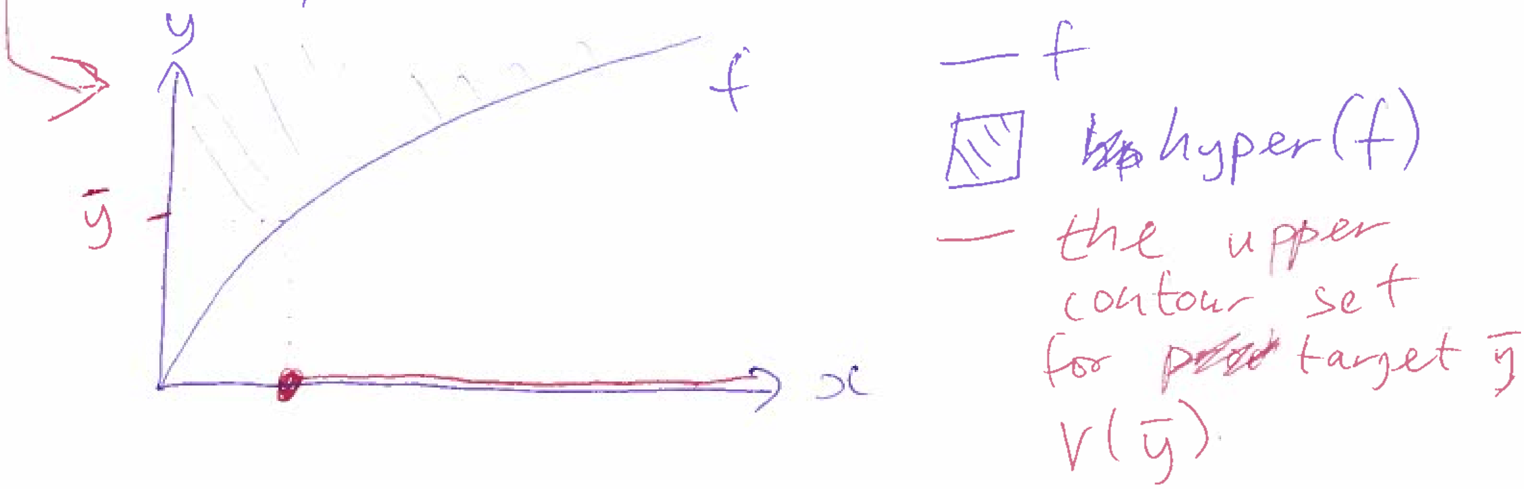
Important: upper contour sets are ~~NOT~~ NOT hypergraphs.

Back to Appendix D.

Def $f: X \to R$ is a quasi-convex function if all its lower contour sets are ~~quasi~~ convex sets. Similarly, f is quasi-concave if all its upper contour sets are convex sets.

Theorem $f: X \to R$ is quasi-convex if and only if
1. X is a convex set, and
2. for all $x, x' \in X$ and all $a \in (0,1)$,
$$f(ax + (1-a)x') \le \max\{f(x), f(x')\}.$$

Hypergraphs vs upper contour sets



Theorem If f is a convex function, then f is a quasi-convex function.

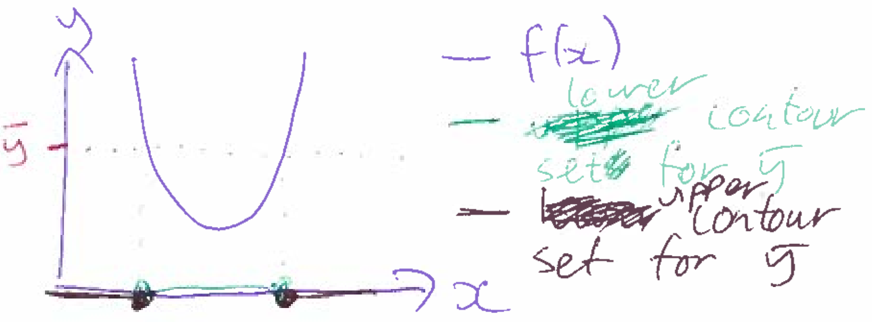
Back to production functions

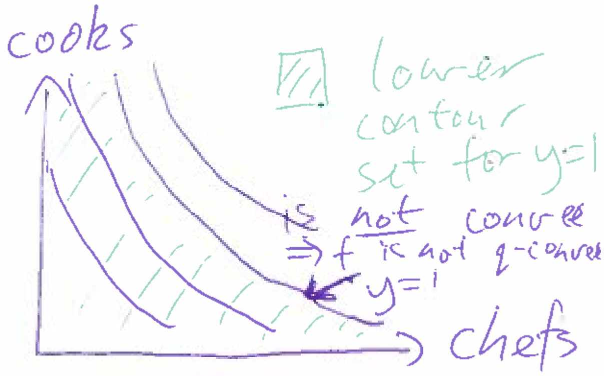
Cont'd assume:

* f is a quasi-concave production function, i.e. each upper contour set $V(y)$ is convex.

Q: Are quasi-concave functions also quasi-convex?

A: Normally, no.





f is quasi-concave, but <u>not</u> quasi-convex.

2.2 Profit maximisation

$p \in \mathbb{R}_+$ ~~out pri~~ output price

$w \in \mathbb{R}^{N-1}_+$ input prices

Firm's <u>profit function</u> is:

$$\pi(p; w) = \max_{x \in \mathbb{R}^{N-1}_+} \underbrace{pf(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{expenditure}} \qquad w \cdot x = \sum_{n=1}^{N-1} w_n x_n$$

~~inputs~~

$$= pf(x(p;w)) - w \cdot x(p;w).$$

the factor demand function

⊛ <u>Aside</u>: If f is strictly concave, then there is only one optimal choice per $(p;w)$.

<u>First-order condition w.r.t. x_i:</u>

$$p \frac{\partial f(x)}{\partial x_i} = w_i$$

$$\frac{\text{FOC } x_i}{\text{FOC } x_j}: \quad \underbrace{\frac{\frac{\partial f(x)}{\partial x_i}}{\frac{\partial f(x)}{\partial x_j}}}_{\text{slope of isoquant}} = \underbrace{\frac{w_i}{w_j}}_{\text{slope of "prices"}}.$$

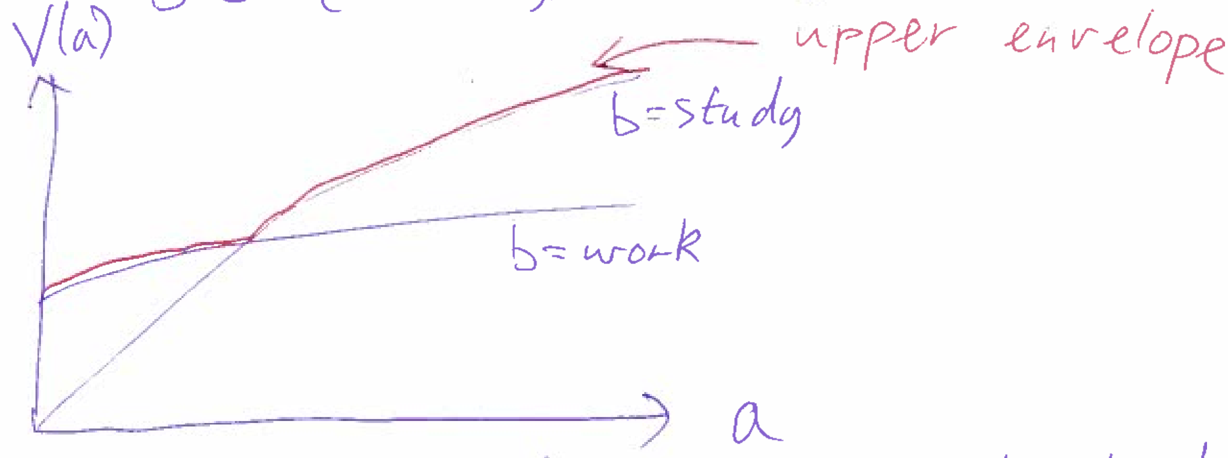
2.3 Upper Envelopes and Value Functions

Problem: $\dfrac{\partial \pi(p, w)}{\partial p}$?

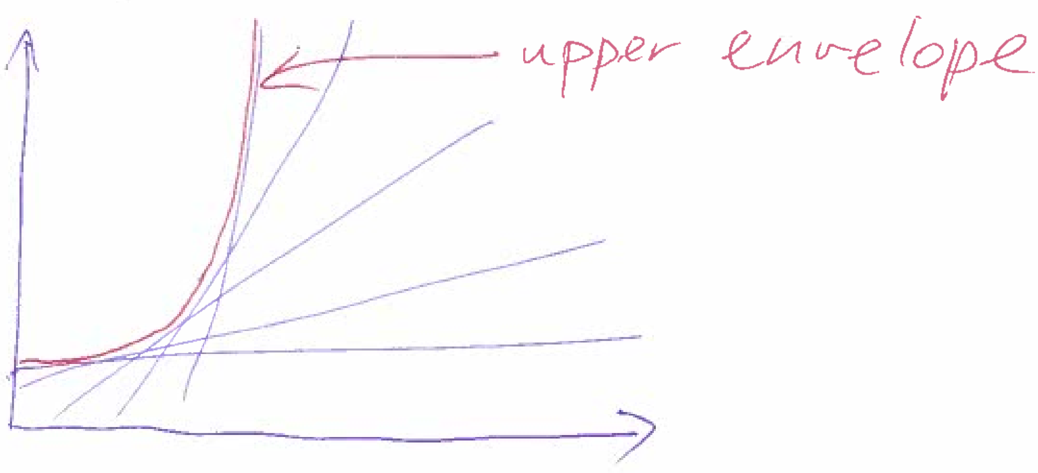
How do we differentiate a max?

$$\underbrace{V(\underbrace{a}_{\text{state variable}})}_{\text{value function}} = \max_{\underbrace{b}_{\text{choice variable}}} \underbrace{v(a, b)}_{\text{objective function}} = v(a, \overbrace{b(a)}^{\text{policy function}})$$

Eg if $b \in \{\text{study}, \text{work}\}$, a assets.



If $b \in \mathbb{R}$, can look more complicated



Envelope Theorem

Let $v: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be a differentiable function, and let $V(a) = \max_b v(a,b)$ be its value function (upper envelope), and $b(a)$ be its policy function.
If V is differentiable, then

$$V'(a) = \left.\frac{\partial v(a,b)}{\partial a}\right|_{b=b(a)}$$

or alternative notation,

$$V'(a) = v_a(a, b(a)).$$