3.4 continued

Ugly formula:

$$x_i(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}.$$

We are stuck!

Terminology:

* **normal good:** good x_i is normal at (p^*, m^*) if demand increases after a wealth increase, i.e. $\frac{\partial}{\partial m} x_i(p^*, m^*) > 0$.
* **inferior good:** good x_i is inferior at (p^*, m^*) if $\frac{\partial}{\partial m} x_i(p^*, m^*) < 0$.

  Note: if x_i is inferior for all m, then the consumer never consumes any of it!
* **Giffen good:** x_i is a Giffen good at (p^*, m^*) if demand increases when the price goes up, i.e. $\frac{\partial x_i(p^*, m^*)}{\partial p_i} > 0$.
* **Substitutes:** x_i and x_j are substitutes at (p^*, m^*) if a price increase in one leads to a consumption increase in the

other, i.e. $\dfrac{\partial x_i(p^*, m^*)}{\partial p_j} > 0.$

$$\| $$

$$\frac{\partial x_j(p^*, m^*)}{\partial p_i}$$

~~special case of: if $f: \mathbb{R}^2 \to \mathbb{R}^2$ is twice differentiable then ∂f~~

* <u>complements</u>: Goods x_i and x_j are complements at (p^*, m^*) if a price increase of one leads to a consumption decrease of the other, i.e. $\dfrac{\partial x_i(p^*, m^*)}{\partial p_j} < 0.$

3.5 Expenditure Functions

The <u>expenditure function</u> is

"time of your life"

$$\underbrace{e(p, \bar{u})}_{} = \min_{x \in \mathbb{R}^N_+} p \cdot x = p \cdot \underbrace{h(p, \bar{u})}_{}$$

$$\text{s.t. } u(x) \geq \bar{u}$$

~~cheapest way~~ how much does it cost to hit utility target $\bar{u}$

Hicksian demand function

Bellman equation:

$$v(p,m) = \max_{\bar u} \; \bar u$$

$$\text{s.t. } e(p,\bar u) = m.$$

"highest affordable utility target".

<u>Big idea</u>: measure "wealth" using utility, not money — not contaminated by prices.

Applying the envelope theorem gives:

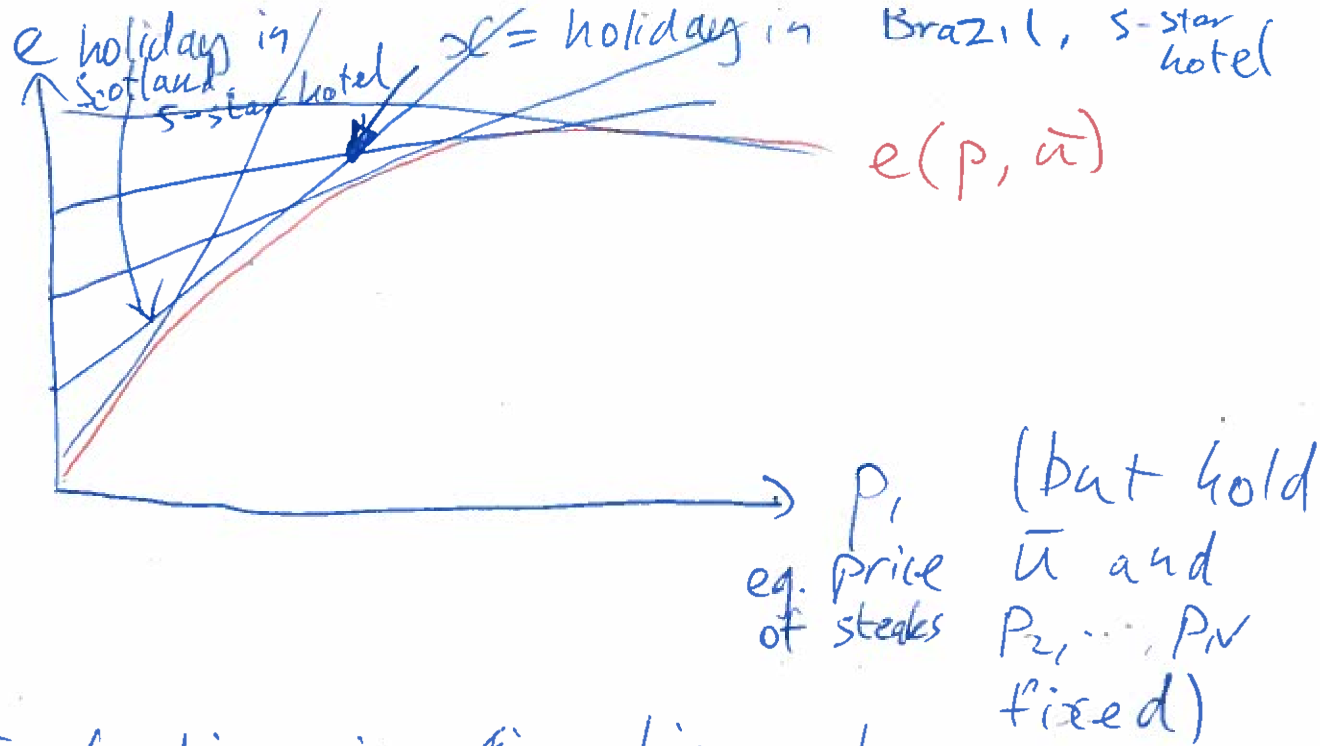
$$\frac{\partial e(p,\bar u)}{\partial p_i} = h_i(p,\bar u)$$

$$= \left[\frac{\partial}{\partial p_i}\left\{p\cdot x - \mu[u(x)-\bar u]\right\}\right]_{\substack{x=h(p,\bar u)\\ \mu=\mu(p,\bar u)}}$$

$$= [x_i]_{x=h(p,\bar u)}$$

$$= h_i(p,\bar u).$$

$$\frac{\partial e(p,\bar u)}{\partial \bar u} = \mu(p,\bar u).$$



* Each line is ~~str~~ lin line, because we hold quantities fixed (eg: 3-star hotel), only changing prices.
* $e(p, \bar{u})$ involves picking the cheapest package holiday — lower envelope.
* $e(p, \bar{u})$ is concave in prices. (Theorem 2.2)

Since $e(\cdot, \bar{u})$ is concave, we deduce: $\dfrac{\partial^2 e(p, \bar{u})}{\partial p_i^{\,2}} \leq 0.$

Therefore: $\dfrac{\partial^2 e(p, \bar{u})}{\partial p_i^{\,2}} = \dfrac{\partial h_i(p, \bar{u})}{\partial p_i} \leq 0.$

If the price of steaks goes up, then ~~they~~ firm would sell a package deal with fewer steaks.

3.6 Slutsky decomposition

Theorem If u, $x(p,m)$ and $h(p,\bar{u})$ are differentiable, then

$$\underbrace{\frac{\partial x_i(p,m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p,\bar{u})}{\partial p_j}\right]_{\bar{u}=v(p,m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p,m)}_{\text{wealth lost}}\,\frac{\partial x_i(p,m)}{\partial m}}_{\text{income effect}}$$

Proof $h(p,\bar{u}) = x(p, e(p,\bar{u}))$.

$$\Rightarrow \quad h_i(p,\bar{u}) = x_i(p, e(p,\bar{u}))$$

$$\Rightarrow \quad \frac{\partial h_i(p,\bar{u})}{\partial p_j} = \left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m}\frac{\partial e(p,\bar{u})}{\partial p_j}\right]_{m=e(p,\bar{u})}$$

LHS

$$\Rightarrow \frac{\partial h_i(p,\bar{u})}{\partial p_j} = \left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m}\, h_j(p,\bar{u})\right]_{m=e(p,\bar{u})}$$

$$\frac{\partial x_i(p,m)}{\partial p_j} = \left[\frac{\partial h_i(p,\bar{u})}{\partial p_j} - \frac{\partial x_i(p,m)}{\partial m} h_j(p,\bar{u})\right]_{\bar{u}=v(p,m)}$$

$$= \left[\frac{\partial h_i(p,\bar{u})}{\partial p_j}\right]_{\bar{u}=v(p,m)} - \frac{\partial x_i(p,m)}{\partial m} x_j(p,m). \qquad \square$$

Chapter 4 Equilibrium

4.1 Economies

Def Pure exchange economy

with N goods and H households consists of:
number $|H|$

* a utility function $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ for each household $h \in H$, set
* an endowment $e_h \in \mathbb{R}^N_+$ for each household $h \in H$.

Def An allocation x specifies each household's consumption $x_h \in \mathbb{R}^N_+$.

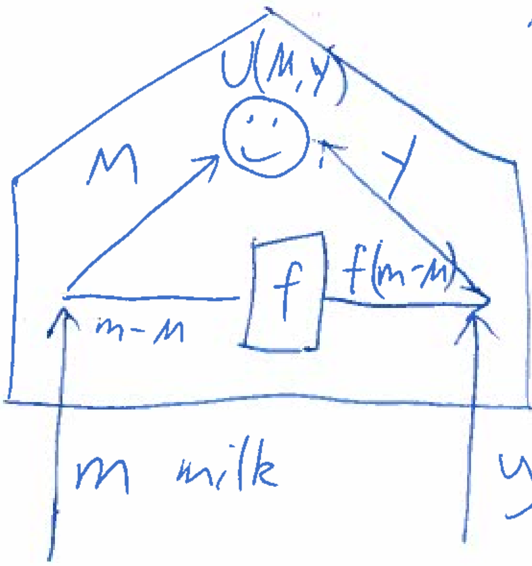
Def An allocation x is feasible if

$$\sum_{h \in H} x_h = \sum_{h \in H} e_h.$$

$$\underbrace{\sum_{h \in H} x_{hn}}_{\text{demand}} = \underbrace{\sum_{h \in H} e_{hn}}_{\text{supply}} \quad \text{for all } n.$$

Aside: home production

M, Y "consumed" quantities



$(-y) = f(m-M) - Y$

"traded"

(<0 if supplying)

Police only sees (m, y)

$$\underbrace{u(m, y)}_{\text{police's observed utility}} = \max_{M, Y} \underbrace{U(M, Y)}_{\text{actual utility}}$$

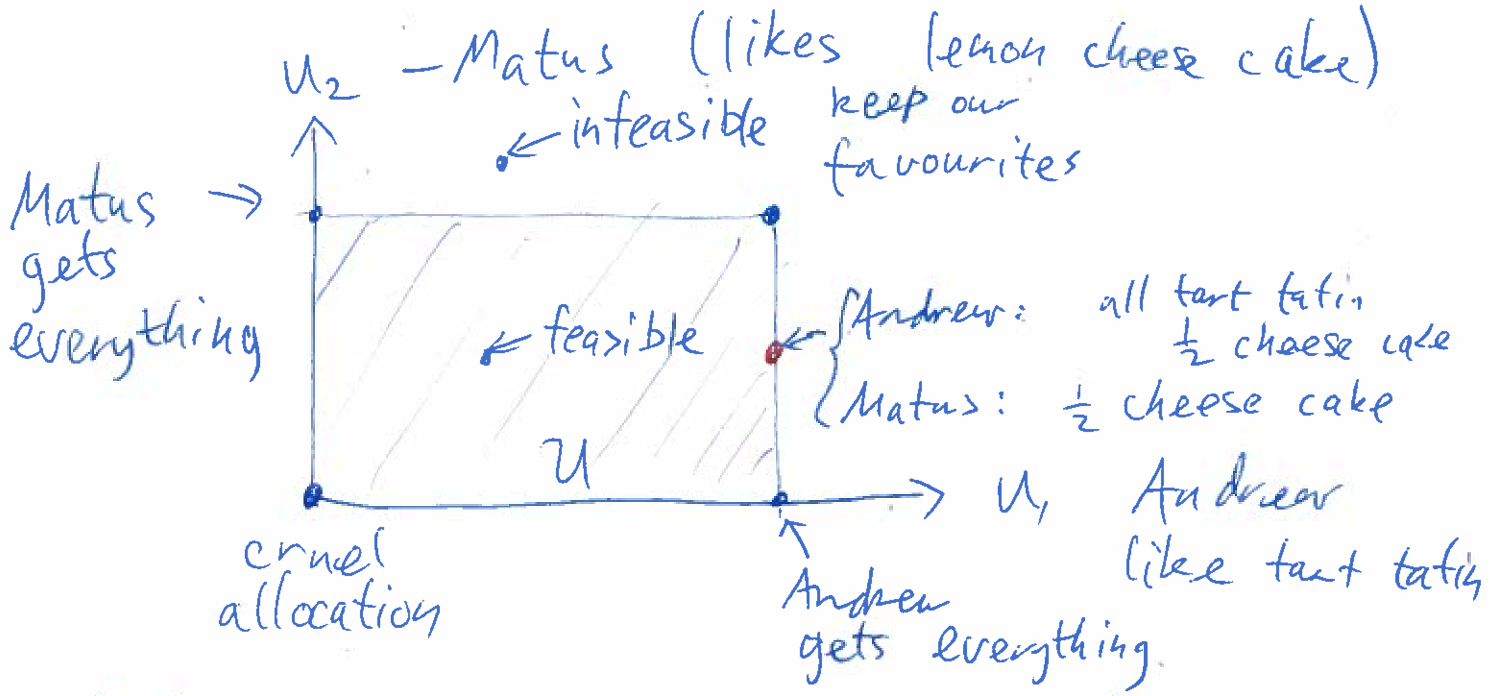
$$\text{s.t. } \underbrace{Y}_{\text{yoghurt consumed}} = \underbrace{f(m-M)}_{\text{yoghurt produced}} + \underbrace{y}_{\text{yoghurt purchased/received}}$$

4.2 Efficient allocations

Def The utility possibility set is

$$\mathcal{U} = \{ (u_h(x_h))_{h \in H} : x \text{ is feasible} \}$$

$$= \{ (u_h(x_h))_{h \in H} : x_h \in \mathbb{R}^N_+, \ \sum_{h \in H} x_h = \sum_{h \in H} e_h \}.$$



We can accommodate <u>free disposal</u> if we replace the feasibility constraint with

$$\sum_{h \in H} x_{hn} \leq \sum_{h \in H} e_{hn} \quad \text{for all } n.$$

<u>Def</u> A vector of utilities $u \in R^H$ <u>Pareto dominates</u> another vector of utilities $u' \in R^H$ if

* $u_h \geq u'_h$ for all households $h \in H$,

and

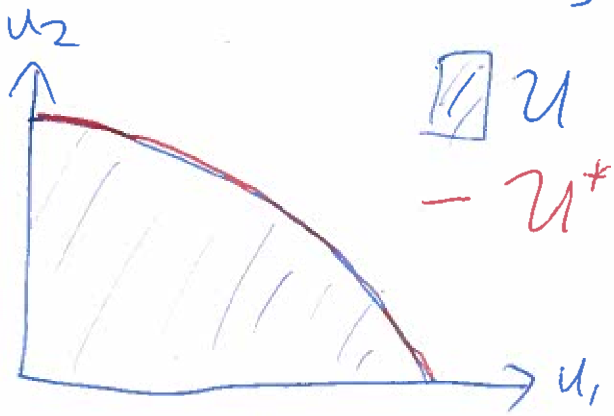
* $u_h > u'_h$ for some household $h \in H$.

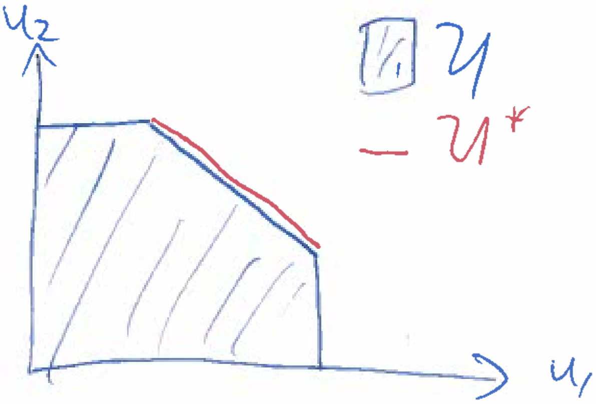
at least one

<u>Def</u> Given a utility possibility set $\mathcal{U}$, a utility vector $u \in R^H$ is <u>efficient</u> if

* u is feasible, i.e. $u \in \mathcal{U}$, and
* for all $u' \in \mathcal{U}$, u' does not Pareto dominate u.

<u>Def</u> The <u>Pareto frontier</u> of $\mathcal{U}$, ~~is~~ denoted $\mathcal{U}^*$, is the set of efficient utility vectors in $\mathcal{U}$.





<u>Def</u> A <u>social welfare function</u> is any function $W: \mathbb{R}^H \to \mathbb{R}$.

<u>Theorem</u> Let $\mathcal{U} \subseteq \mathbb{R}^H$ be a utility possibility set, and $W: \mathbb{R}^H \to \mathbb{R}$ be a strictly increasing social welfare function. If $u \in \mathcal{U}$ maximises social welfare, i.e.

"argument" the set of optimal choices for the optimisation problem

$$u \in \arg\max_{\hat{u} \in \mathcal{U}} W(\hat{u})$$

then u is Pareto efficient, i.e. $u \in \mathcal{U}^*$.

4.3 Equilibrium

Def Consider a pure-exchange economy, (u_h) and (e_h). (We could write (u, e).) We say that (x^*, p^*), consisting of an allocation $x^* \in \mathbb{R}_+^{NH}$ and prices $p^* \in \mathbb{R}_+^N$ is a *pure exchange equilibrium* if

* $x_h^* \in \underset{x_h \in \mathbb{R}_+^N}{\arg\max}\ u_h(x_h)$

$$\text{s.t.} \quad p^* \cdot x_h \leq p^* \cdot e_h \quad \text{or } =$$

for all $h \in H$, and

* all markets clear, i.e.

$$\sum_h x_h^* = \sum_h e_h .$$

4.4 Characterising Equilibria

Def ~~Excess demand~~ The *excess demand function* of a pure-exchange economy is

$$z(p) = \sum_{h \in H} \left[x_h(p) - e_h \right]$$

where $z : \mathbb{R}_+^N \to \mathbb{R}^N$ and $x_h(p)$ is each household's demand function.

eg: if $n=1$ is hotel rooms and $z_1(p) > 0$, then it's not possible squeeze everyone in. If $z_1(p) < 0$, then there are vacancies, at that price level.

~~[illegible]~~ (x^*, p^*) is an equilibrium if and only if $x_h^* = x_h(p^*)$ and

$$z(p^*) = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Theorem (Walras' law) Consider a pure-exchange economy (u, e) with strictly increasing utility functions. Let z be its excess demand function.

(i) The excess demand satisfies

$$p \cdot z(p) = 0 \text{ for all } p \in \mathbb{R}^N_{++}.$$

(ii) If $N-1$ markets clear (i.e. supply = demand in those markets) for $p \in \mathbb{R}^N_{++}$, then all markets clear.

(iii) For every $p \in \mathbb{R}^N_{++}$, $z(p) \neq 0$ if and only if there is excess demand in some market i and excess supply in some market j.

<u>Proof</u> (i) Since each household h exhausts its budget constraint,

$$p\cdot(x_h(p)-e_h)=0 \text{ for all } h\in H.$$

Summing up over all households gives

$$\sum_{h\in H} p\cdot(x_h(p)-e_h) = p\cdot\Big[\sum_{h\in H} x_h(p)-e_h\Big]$$

$$= p\cdot z(p) = 0.$$

"Let's make an innocuous assumption."

(ii) Without loss of generality, assume the first $N-1$ markets clear at price p. Then $z_j(p)=0$ for $j\in\{1,\dots,N-1\}$.

Therefore $p_j z_j(p)=0$ " " " "

$$\Rightarrow \sum_{j=1}^{N-1} p_j z_j(p) = 0.$$

Now, $\underbrace{p\cdot z(p)}_{=0\ (\text{from (i)})} - \underbrace{\sum_{j=1}^{N-1} p_j z_j(p)}_{=0} = p_N z_N(p)=0$

Since $p_N>0$, we conclude $z_N(p)=0$.

(iii) <u>Boring bit</u>: if $z_i(p)>0$ or $z_j(p)<0$ then clearly $z(p)\neq 0$.

<u>Interesting bit</u>: ~~If~~ Suppose $z(p)\neq 0$. Suppose for the sake of contradiction that there is excess demand in market i but no excess supply in any other

market, i.e. $z_i(p) > 0$ and $z_j(p) \geq 0$ for all j. Then $p \cdot z(p) > 0$, contradicting (i). Our supposition was false — we just ruled out excess demand without excess supply. A similar proof rules out excess supply without excess demand. $\square$