Proof (Chain rule proof).

$$V(a) = v(a, b(a)).$$

Lagrangian trick:

$$V(a) = v(a, b(a)) + \lambda(a) w(a, b(a))$$
$$= L(a, b(a), \lambda(a)).$$

Differentiating gives:

$$V'(a) = \left[\frac{\partial L(a,b,\lambda)}{\partial a} + \overbrace{\frac{\partial L(a,b,\lambda)}{\partial b}}^{=0} b'(a) + \overbrace{\frac{\partial L(a,b,\lambda)}{\partial \lambda}}^{=0} \lambda'(a) \right]_{b=b(a), \lambda = \lambda(a)}$$

By the FOC, the red circled bit is 0.

start of lecture 3

The green circled bit is

$$\frac{\partial L(a,b,\lambda)}{\partial \lambda} = w(a,b)$$

which equals zero at $(a, b(a))$ since we assumed the constraint binds at a.

We conclude

$$V'(a) = \left. \frac{\partial L(a,b,\lambda)}{\partial a} \right|_{b=b(a), \lambda=\lambda(a)}$$
$$= \left[\frac{\partial v(a,b)}{\partial a} + \lambda \frac{\partial w(a,b)}{\partial a} \right]_{b=b(a), \lambda=\lambda(a)}$$

$\square$

Recall the cost function is

$$c(y; w) = \min_{x \in \mathbb{R}^{N-1}_+} w \cdot x$$
$$\text{s.t. } f(x) \geq y.$$

The Lagrangian is

$$L(y, w; x; \lambda) = w \cdot x - \lambda [f(x) - y].$$

The constrained envelope theorem implies (?) that

$$\frac{\partial c(y; w)}{\partial y} = \left\{ \frac{\partial}{\partial y} \left[w \cdot x - \lambda [f(x) - y] \right] \right\}_{\substack{x = x(y;w) \\ \lambda = \lambda(y;w)}}$$

$$= \{\lambda\}_{\substack{x = x(y;w) \\ \lambda = \lambda(y;w)}}$$

$$= \lambda(y; w).$$ ← a good way to understand Lagrange multipliers.

and

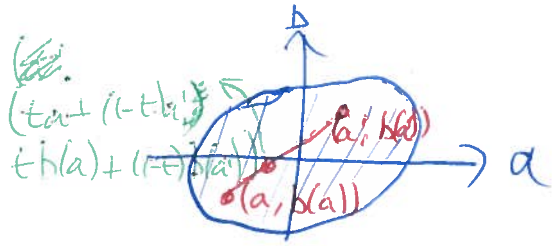
$$\frac{\partial c(y; w)}{\partial w_i} = \left\{ \frac{\partial}{\partial w_i} \left[w \cdot x - \lambda [f(x) - y] \right] \right\}_{\substack{x = x(y;w) \\ \lambda = \lambda(y;w)}}$$

$$= \{x_i\}_{\substack{x = x(y;w) \\ \lambda = \lambda(y;w)}}$$

$$= x_i(y; w).$$ ← Shephard's Lemma

Theorem 2.6

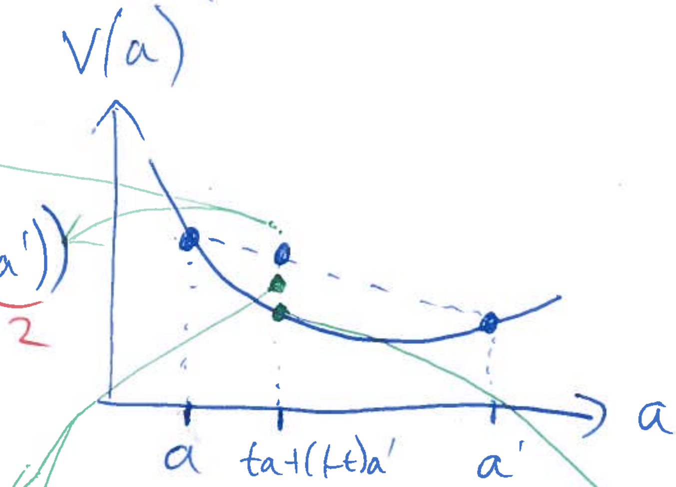
Let



$$V(a) = \min_b v(a,b)$$
$$\text{s.t. } w(a,b) \ge 0.$$

If v is a convex function and w is a quasi-concave function then V is a convex function.

Proof



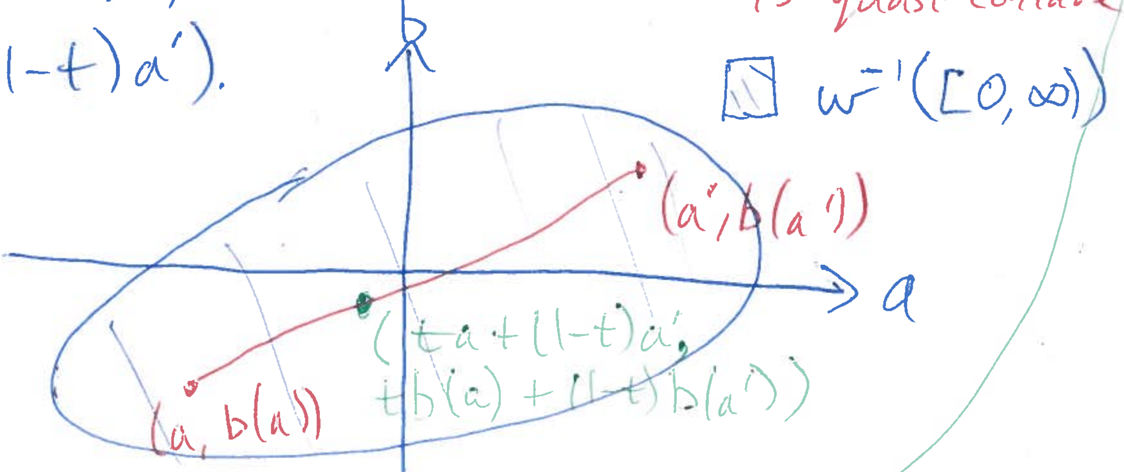
$$tV(a) + (1-t)V(a')$$
$$= t\, v(\underbrace{a, b(a)}_{\text{point 1}}) + (1-t)\, v(\underbrace{a', b(a')}_{\text{point 2}})$$
$$\ge \cancel{t\, v(a, b)}\; v(\underbrace{t\,(a, b(a)) + (1-t)\,(a', b(a'))}_{\text{intermediate point}})$$

← since v is convex

$$= v(ta + (1-t)a', tb(a) + (1-t)b(a'))$$

← feasible choice since w is quasi-concave

$$\ge v(ta + (1-t)a', b(ta + (1-t)a'))$$
$$= V(ta + (1-t)a').$$



Theorem 2.7 If the production function f is concave, then the cost function is convex in the output target y, i.e.

$c(\cdot\,; w)$ is convex ~~for all w.~~

Not saying c is a convex function.

Proof

Objective function:

$$(y; x) \mapsto \underbrace{w \cdot x}_{\text{linear } (\Rightarrow \text{ convex})}$$

Constraint:

$$(y; x) \mapsto \underbrace{\underbrace{f(x)}_{\text{concave}} + \underbrace{-y}_{\text{linear } (\Rightarrow \text{ concave})}}_{}$$

$\Rightarrow$ concave

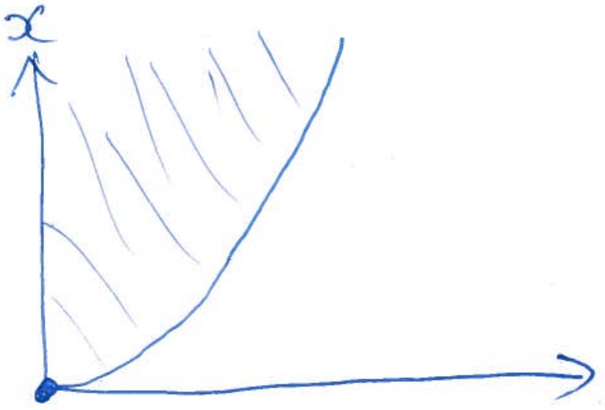
$\Rightarrow$ quasi-concave

By theorem 2-6,

$$c(y) = \min_{x} \; w \cdot x \quad \text{s.t. } f(x) - y \geq 0$$

is a convex function. $\square$

eg: with only one input $x \in \mathbb{R}_+$



$w^{-1}([0,\infty))$ $= \{(y,x) : w(y,x) \in [0,\infty)\}$

where $w(y,x) = f(x) - y$.

"x meets the production target y"

Chapter 3

3.1 Utility functions

Def Consider two possible choices $x, y \in \mathbb{R}^N_+$. If a consumer weakly prefers x to y, we write $x \succsim y$. If the preference is strict, we write $x \succ y$, shorthand for $x \succsim y$ but $y \not\succsim x$. If $x \succsim y$ and $y \succsim x$, then we say the consumer is indifferent between x and y.

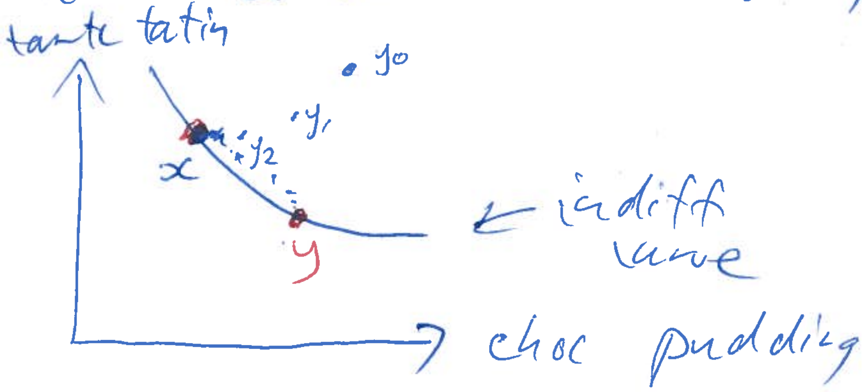
Def A utility function is a function $u: \mathbb{R}^N_+ \to \mathbb{R}$. We say that u represents the preferences $\succsim$ if for all choices $x, y \in \mathbb{R}^N_+$,

$$x \succsim y \iff u(x) \geq u(y).$$

~~Def~~ Possible assumptions about preferences:

* complete: for all x, y, either $x \succsim y$ or $y \succsim x$

* reflexive: $x \succsim x$ for all x
* transitive: if $x \succsim y$, $y \succsim z$, then $x \succsim z$.
* (*) continuity: all upper and lower contour sets are closed in $(\mathbb{R}^N_+, d_2)$.



Suppose $y_n \to y$ and $y_n \succsim x$ then $y \succsim x$.

Theorem 3.1 (*)

Consider a preference relation $\succsim$. Then there exists a continuous utility function u that represents $\succsim$ if and only if $\succsim$ is complete, reflexive transitive and continuous.

Theorem 3.2 Consider a preference relation $\succsim$, and any strictly increasing function $g: \mathbb{R} \to \mathbb{R}$, e.g. $g(x) = 2x$, or $g(x) = \sqrt{x}$. Then $u: \mathbb{R}^N_+ \to \mathbb{R}$ represents $\succsim$ if and only if $v(x) = g(u(x))$ represents $\succsim$. Utility is ordinal, not cardinal

Another possible assumption:

* ~~convexity~~: all upper contour sets, i.e. sets of the form $U(y) = \{x \in \mathbb{R}^L_+ : x \succsim y\}$, are convex sets.

Note: if $\succsim$ is convex, and u represents $\succsim$, then u is quasi-concave.

3.2 Time preference

Suppose there are T time periods. ~~Let~~ If J is a subset of $\{1, \ldots, T\}$ and $x \in X^T$ then $x_J = (x_j)_{j \in J}$.

For example if $x = (2, 4, 5)$ then $x_{\{2,3\}} = (4, 5)$.

I write $-J$ to mean $\{1, \ldots, T\} \setminus J$.

Given two choices $x, y \in$ ~~X~~ X^T, $z = (x_J, y_{-J})$ represents ~~x~~ consuming x in the J time periods and y otherwise.

If $x = (1, 2, 3)$ and $y = (4, 5, 6)$ and $J = \{2\}$ then $(x_J, y_{-J}) = (4, 2, 6)$.

Def (Time separable)

Consider the choice set X^T, where $X = \mathbb{R}^N_+$, i.e. T time periods and N goods. A preference relation $\succsim$ is <u>time separable</u> if for any $x, y \in X^T$ that coincide in the J periods (i.e. $x_J = y_J$), the preference is unaffected by simultaneously changing the J choices to z_J, i.e.

$$x \succsim y \iff (\cancel{x_J}\, z_J, x_{-J}) \succsim (z_J, y_{-J})$$

for all z_J.

<u>Example</u> One good ($N=1$) methamphetamine, and four days ($T=4$), ~~also~~ only two quantities $\{0,1\}$. The choice set is X^T where $X = \{0,1\}$. $|X^T| = 16$.

$x = 0000$
$y = 0011$
$x' = 0100$ ← "cold turkey"
$y' = 0111$

Most (?) people prefer $x \succsim y$, and $y' \succsim x'$.

These preferences are <u>NOT</u> time separable.

Set $J = \{1,2\}$ and $z_J = y'_J = 01$.

Then $x_J = y_J$ and $x \succsim y$ so time separability would imply

$$x' = (z_J, x_{-J}) \succsim (z_J, y_{-J}) = y'.$$

But this is false.

<u>Theorem 3.3 A</u> If the preferences $\succsim$ over X^T are complete, reflexive, transitive, continuous, time-separable, strictly increasing, and $T \geq 3$, then there exist continuous utility functions $u_1, \ldots, u_T$ such that $\precsim$ is represented by

$$U(x) = \underbrace{u_1(x_1) + u_2(x_2) + \cdots + u_T(x_T)}_{\text{additively separable utility}}.$$

If $\underbrace{U(x) = u(x_1) + \beta u(x_2) + \cdots + \beta^{T-1} u(x_T)}_{\text{discounted utility.}}$

e.g. Cake-eating problem:

$$\underbrace{V_t(k_t)}_{\substack{\text{cake size} \\ \text{at start of} \\ \text{time } t}} = \max_{\underbrace{x_t, \dots, x_T}_{\text{consumption}} \geq 0} u_t(x_t) + \cdots + u_T(x_T)$$

$$\text{s.t. } x_t + \cdots + x_T = k_t.$$

Bellman equation

$$V_t(k_t) = \max_{x_t, k_{t+1} \geq 0} u_t(x_t) + V_{t+1}(k_{t+1})$$

$$\text{s.t. } x_t + k_{t+1} = k_t$$

$$\text{if } t < T$$

and ~~$V_T(k_T)$~~ $V_T(k_T) = u_T(k_T)$.

<u>Proof of principle of optimality:</u>

$$V_t(k_t) = \max_{x_t, \dots, x_T \geq 0} u_t(x_t) + \cdots + u_T(x_T)$$

$$\text{s.t. } x_t + \cdots + x_T = k_t$$

$$= \max_{\substack{x_t, \dots, x_T \geq 0, \\ k_{t+1} \geq 0}} u_t(x_t) + \cdots + u_T(x_T)$$

$$\text{s.t. } x_t + \cdots + x_T = k_t$$

$$\text{and } x_t + k_{t+1} = k_t$$

$$= \max_{\substack{x_t, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} \left[\max_{x_{t+1}, \dots, x_T \geq 0} u_t(x_t) + \cdots + u_T(x_T) \quad \text{s.t. } x_t + \cdots + x_T = k_t \right]$$

split

$$= \max_{\substack{x_t, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} \left[\max_{\substack{x_{t+1}, \cdots, x_T \geq 0 \\ \text{s.t. } x_{t+1} + \cdots + x_T = k_{t+1}}} u_t(x_t) + \cdots + u_T(x_T) \right]$$

$$= \max_{\substack{x_t, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} u_t(x_t) + \underbrace{\left[\max_{\substack{x_{t+1}, \cdots, x_T \geq 0 \\ \text{s.t. } x_{t+1} + \cdots + x_T = k_{t+1}}} u_{t+1}(x_{t+1}) + \cdots + u_T(x_T) \right]}_{= V_{t+1}(k_{t+1})}$$

$$= \max_{\substack{x_t, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} u_t(x_t) + V_{t+1}(k_{t+1}).$$

3.3 Utility Maximisation

§ Utility max. problem is

$$\underbrace{v(p, m)}_{\substack{\text{value function} \\ \text{("indirect utility)}}} = \max_{x \in \mathbb{R}^N_+} u(x) \quad \text{s.t. } \underbrace{p \cdot x}_{\text{expenditure}} \leq \underbrace{m}_{\text{income}}$$

prices, money

or

$$v^*(p, e) = \max_{x \in \mathbb{R}^N_+} u(x) \quad \text{s.t. } \underbrace{p \cdot x}_{\text{expenditure}} \leq \underbrace{p \cdot e}_{\text{income}}$$

endowment $\in \mathbb{R}^N_+$

The first order condition w.r.t. x_i:

$$\left[\frac{\partial u(x)}{\partial x_i} - \lambda p_i \right]_{\substack{x = x(p,m) \\ \lambda = \lambda(p,m)}} = 0$$

This can be rewritten as

$$\left.\frac{\frac{\partial u(x)}{\partial x_i}}{p_i}\right|_{x=x(p;m)} = \lambda(p;m).$$

and

$$\frac{\frac{\partial u(x)}{\partial x_i}}{p_i} = \frac{\frac{\partial u(x)}{\partial x_j}}{p_j}.$$

3.4 Consumer's value and policy functions

Obvious: $p\downarrow$ or $m\uparrow$ then $v\uparrow$.

Apply the envelope theorem to the value function:

① $$\frac{\partial v(p,m)}{\partial p_i} = -\lambda(p,m)\, x_i(p,m)$$

② $$\frac{\partial v(p,m)}{\partial m} = \lambda(p,m).$$

$-\frac{①}{②}$: $$x_i(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}.$$