Micro 1

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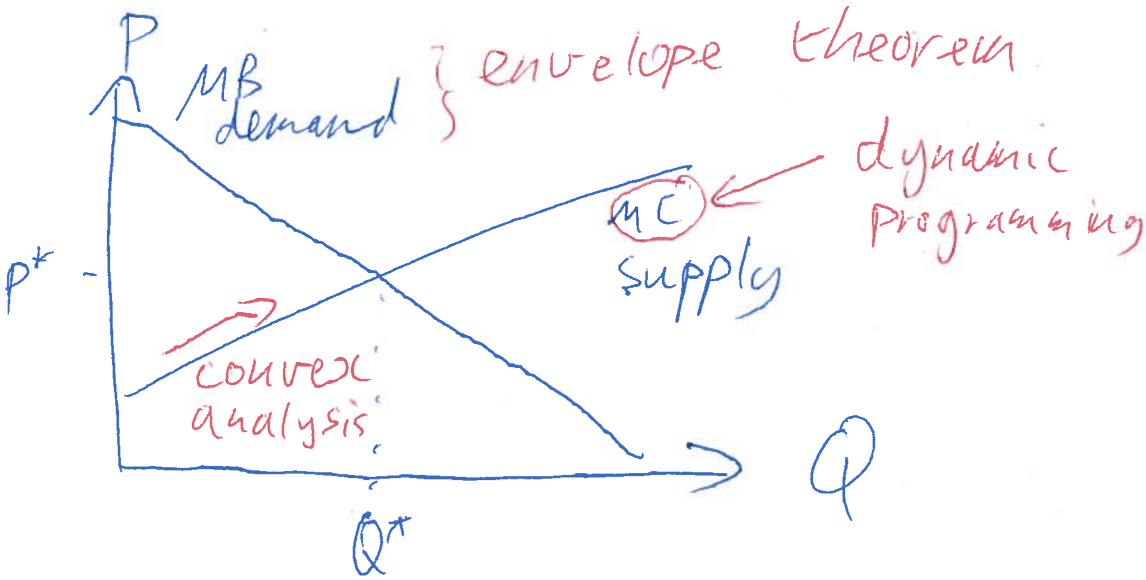
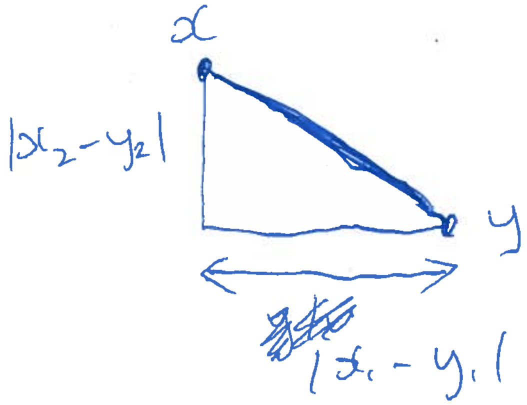


Bonus: Metric Spaces (CI)

Pythagoras: $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$



Def (X, d) is a metric space if it consists of a point set X and distance metric $d: \underbrace{X \times X}_{\text{two points in}} \to \underbrace{\mathbb{R}_{+}}_{\text{one number out}}$ ← no negative numbers

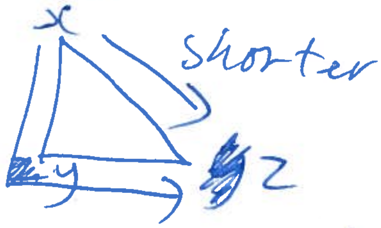
and ~~so~~ ~~that~~ it satisfies three properties:

(i) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$,

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$, and

(iii) $d(x,z) \leq \underbrace{d(x,y) + d(y,z)}_{\text{indirect route via } y}$ for all $x, y, z \in X$

– traditionally called "the triangle inequality"



or "no-shortcut property"

Examples: eg: $\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\} = \{(0,0), (0,1), (0.5, 0.2), \dots\}$

* $(\mathbb{R}^n, d_1)$ where $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, "Manhattan metric",
* $(\mathbb{R}^n, d_2)$ where $d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ "Euclidean metric" (Pythagoras!)
* (X, d_∞) where $X = \{f : [0,1] \to [0,1]\}$ and $d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)|$

supremum – like maximum

Problem with max:

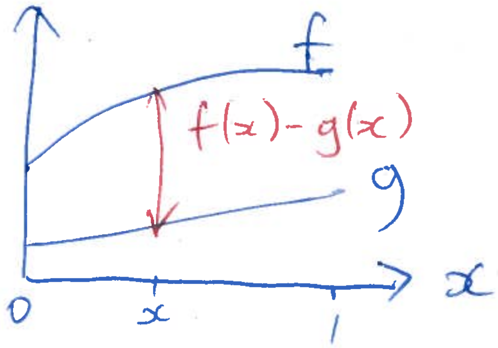
Q: What is max $[0,1)$?

$\quad\quad\quad\quad\quad\quad \| \{x \in \mathbb{R} : x \geq 0 \text{ and } x < 1\}$.

A: There is no maximum!

Solution: $\sup [0,1) = 1$.

"smallest number that is at least as big as everything in the set."



C2 Convergence

Def A sequence in the set X is any function with domain $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and co-domain X. Usually $f : \mathbb{N} \to X$.

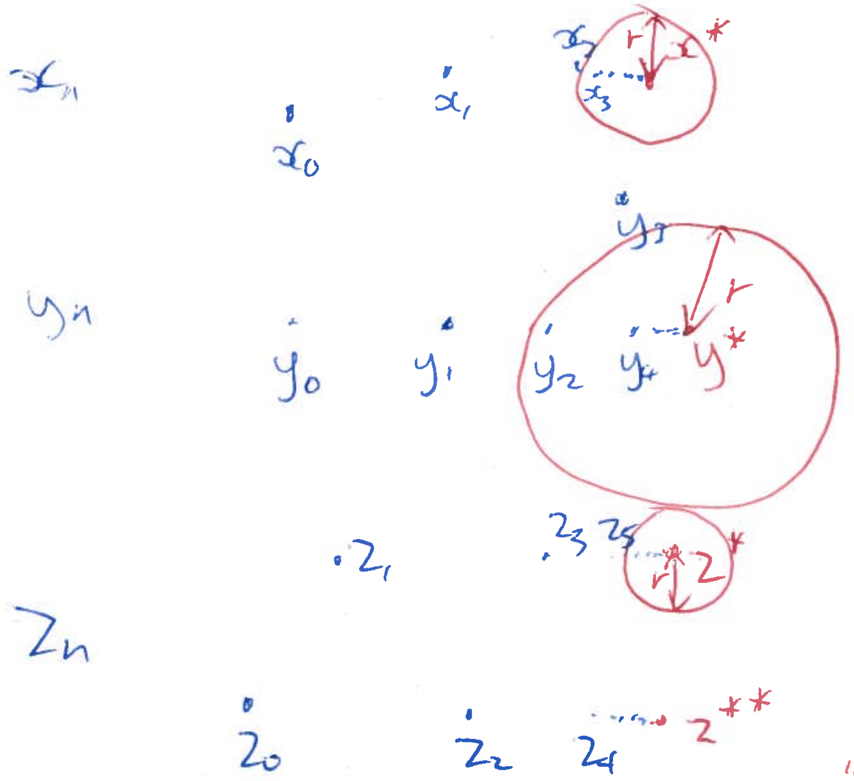
But special notation: $x_0, x_1, x_2, \dots$

or $\{x_n\}_{n=0}^{\infty}$ or x_n.

eg: $x_n = 2n = 0, 2, 4, 6, 8, \dots$

Def Suppose x_n is a sequence in a metric space (X, d). We say that x_n converges to $x^* \in X$ (or write $x_n \to x^*$) if for every $r > 0$, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x^*) < r \text{ for every } n \geq N.$$

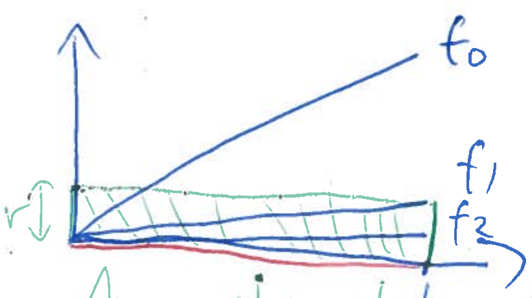


need $N=3$

need $N=4$

NOT a convergent sequence

There is no N that excludes all of the "bottom" points



$f_n(x) = \frac{1}{n} x$

in (X, d_∞) from before

$f_n \to 0$ (in red)

Any function g inside ▨ has $d(0, g) < r$.

2.1 Production functions

N types of goods (and services, etc.)

$N-1$ # inputs

$x \in \mathbb{R}_+^{N-1}$ inputs

$y = f(x)$ output

$f : \mathbb{R}_+^{N-1} \rightarrow \mathbb{R}_+$ production function

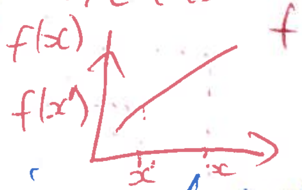
Possible assumptions:

$\underbrace{(0, 0, \dots, 0)}_{N-1}$

* possibility of inaction: $f(0) = 0$
* free disposal (monotonicity):
  If $x \geq x'$ (i.e. if $x_n \geq x_n'$ for all $n \in \{1, \dots, N-1\}$)
  then $f(x) \geq f(x')$.

music: monotone is the same note over and over

maths: monotone increasing means "up and up and up"



We can also assume f is strictly (monontically) increasing, i.e. if $x > x'$ ($x \geq x'$ and for some n, $x_n > x_n'$) then $f(x) > f(x')$.

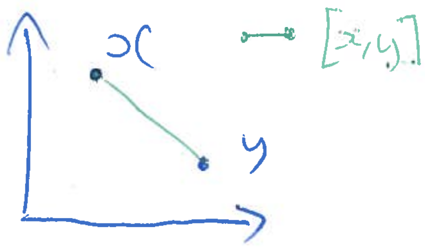
* Smoothness: f is twice continuously differentiable. $\frac{\partial}{\partial x_1} f(x)$ is the marginal productivity of input good 1.

Detour (to generalise ↓VMP, ↓RTS)

D Convex geometry

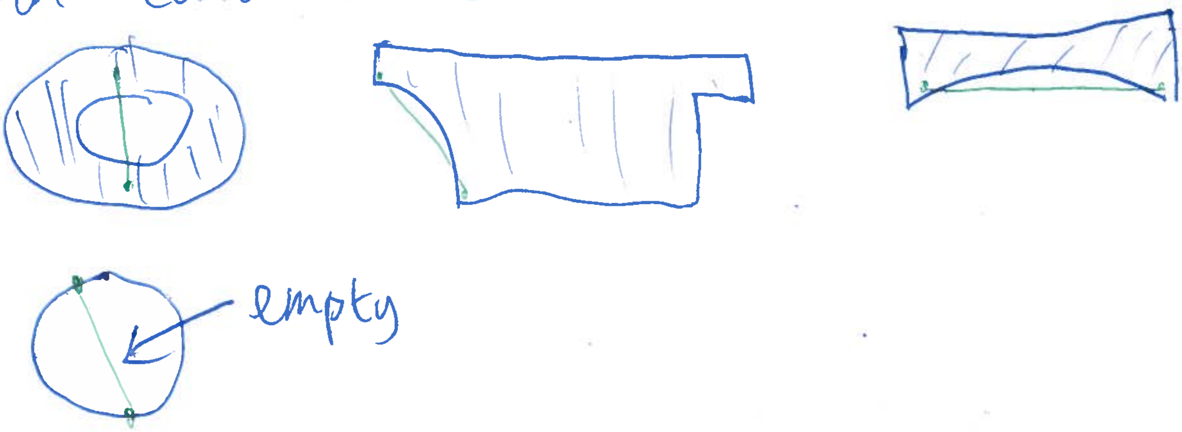
Def A closed interval between $x, y \in \mathbb{R}^n$ is defined as

$$[x,y] = \{\underbrace{ax + (1-a)y}_{\text{convex combination}} : a \in [0,1]\}$$

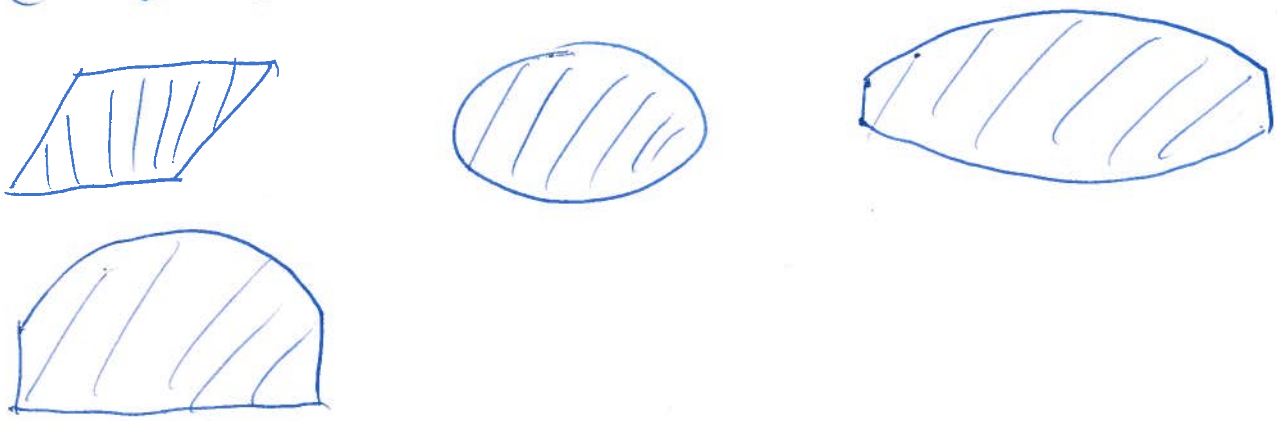


Def $X \subseteq \mathbb{R}^n$ is a convex set if for all $x, y \in X$, the interval $[x,y] \subseteq X$.

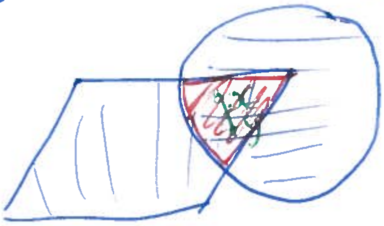
Non-convex sets



Convex sets:



Theorem The ~~intersections~~ intersection of convex sets is convex.



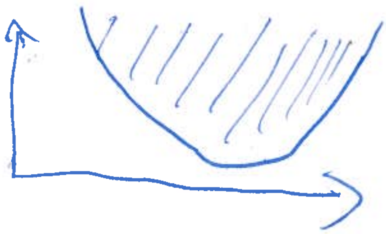
~~con~~ intersection

Proof: Suppose A and B are convex sets. We need to show that for every $x, y \in A \cap B$, the interval $[x,y] \subseteq A \cap B$. Since $x, y \in A \cap B$, we know $x, y \in A$. Since A is a convex set, we know $[x,y] \subseteq A$. Similarly, $[x,y] \subseteq B$. Combining, $[x,y] \subseteq A \cap B$. $\square$

Def $f : X \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}$ is a <u>convex function</u> if its <u>hypergraph</u>

$$\text{hyper}(f) = \{(x,y) : x \in X,\ y \geq f(x)\}$$

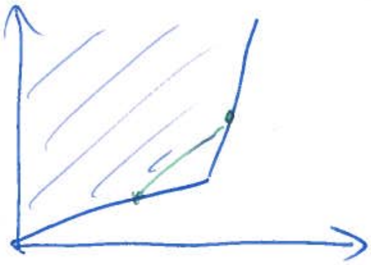
is a convex set.



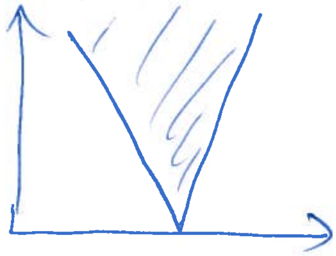
$-f$

$\text{hyper}(f) \subseteq X \times \mathbb{R}$

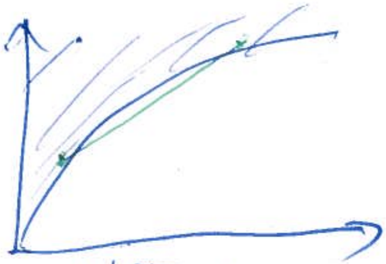
convex



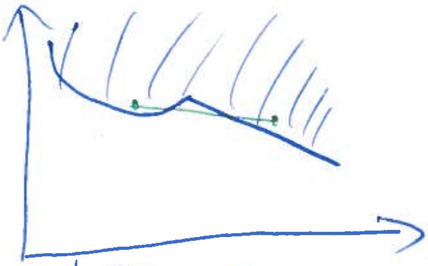
convex



convex

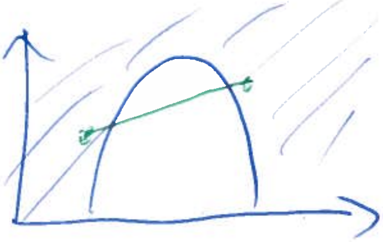


NOT convex



NOT convex

NOT convex

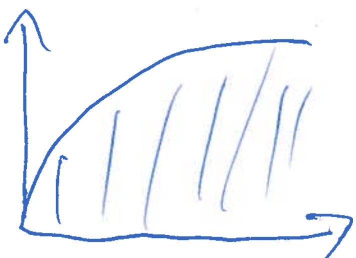


NOT convex

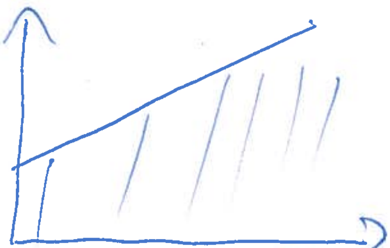
<u>Def</u> $f: X \to \mathbb{R}$ is a <u>concave function</u> if $g(x) = -f(x)$ is a a convex function (or equivalently, its hypograph, $\text{hypo}(f) = \{(x, y) : f(x) \geq y\}$ is a convex set).

Confusion: ① convex functions and sets

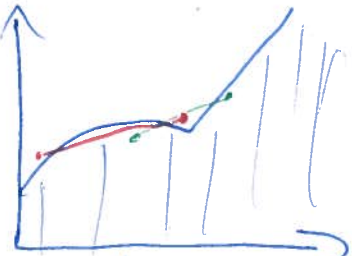
② ~~[illegible]~~ no such thing as a concave set.



concave



concave (and convex)



not concave (and not convex)

Bonus

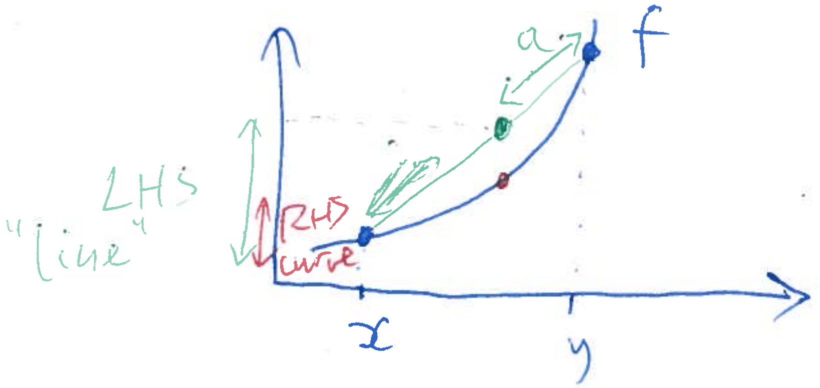
(*) <u>Theorem</u> If $f: X \to \mathbb{R}$ is a convex function and $X \subseteq \mathbb{R}^n$ is an open set, then f is continuous.

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is convex if and only if its derivative f' is (weakly) monotonically increasing

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is convex if and only if $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

<u>Theorem</u> A function $f: X \to \mathbb{R}$ is convex if and only if X is convex and for all $x, y \in X$ and all $\underbrace{a \in (0,1)}_{\text{weight}}$

$$\underbrace{a f(x) + (1-a) f(y)}_{\text{line}} \geq \underbrace{f(a x + (1-a) y)}_{\text{curve}}$$



2.1 Production Functions (again)

* decreasing marginal productivity:
  ~~eg~~ for good 1: $\frac{\partial}{\partial x_1} f(x)$ is decreasing as x_1 increases (and $x_2, \dots, x_{N-1}$ are held fixed)

  $\Leftrightarrow$ $g(x_1) = f(x_1; x_2, \dots, x_{N-1})$ is concave for all $(x_2, \dots, x_{N-1})$.

* increasing returns to scale:
  for all $x \in \mathbb{R}_+^{N-1}$ and all $t \geq 1$,

  $$f(tx) \geq t f(x).$$

eg $t=2$: double inputs   double outputs

* constant ~~RR~~ RTS: for all $x \in \mathbb{R}_+^{N-1}$ and all $t > 0$, $f(tx) = t f(x)$.

* decreasing RTS: for all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$, $f(tx) \leq t f(x)$.
  ~~Assump~~ This assumption is sometimes made to capture non-tradeable resources — but this goes against GE philosophy/motivation.

* concave: f is a concave function.
  - if f is smooth & concave, then f has decreasing marginal productivity
  - if f is concave and has the possibility of inaction, then f has decreasing (constant) RTS.

<u>Proof</u>: Want to show: $f(tx) \leq t f(x)$, ~~that~~ for all $t > 1$. Let $a = \frac{1}{t} \in (0,1)$.

Recall f is concave

$$\Leftrightarrow \underbrace{a f(X) + (1-a) f(X')}_{\text{line}} \leq f(aX + (1-a)X')$$

Let $X = tx$ and $X' = 0$.

~~Pro~~ Inequality becomes: ~~f(0)~~ inaction

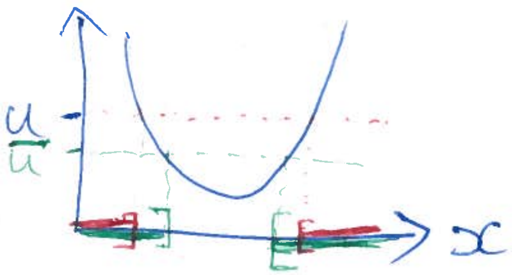
$$\frac{1}{t} f(tx) + \left(1 - \frac{1}{t}\right) f(0) \leq f\left(\frac{1}{t} t x + \left(1 - \frac{1}{t}\right) 0\right)$$

$$\Leftrightarrow \frac{1}{t} f(tx) + 0 \leq f(x)$$

$$\Leftrightarrow f(tx) \leq t f(x). \quad \square$$

Detour: back to convex analysis

<u>Def</u> The <u>upper contour set</u> of a function $f: X \to \mathbb{R}$ at level u is $\{x : f(x) \geq u\}$.



— $\{x : f(x) \geq u\} \subseteq X$

— $\{x : f(x) \geq \bar{u}\}$

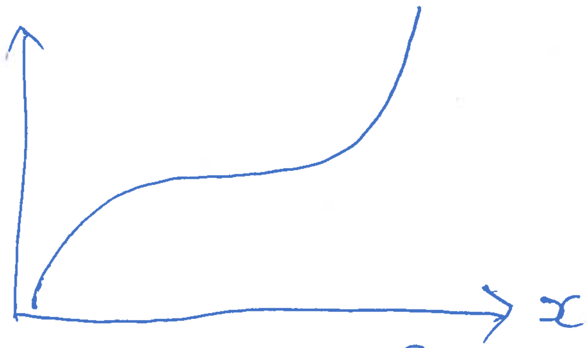
Confusion:

* superficially similar to hypergraphs, but details are totally different!

Similarly, the <u>lower contour set</u> of f at level u is $\{x \in X : f(x) \leq u\}$.

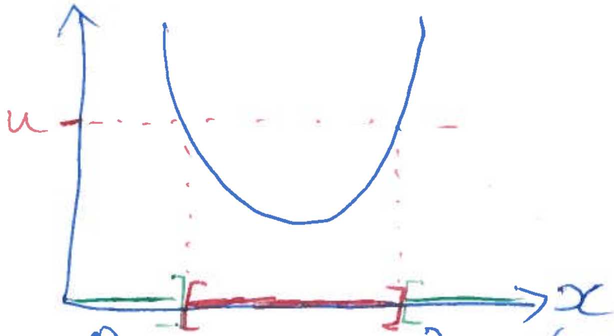
<u>Def</u>: $f : X \to R$ is a <u>quasi-convex function</u> if all of its lower ~~contours~~ contour sets (i.e. at each level) are convex sets.

Similarly, f is a <u>quasi-concave function</u> if its upper contour sets are convex sets.



Quasi-convex? ✓

Quasi-concave? ✓



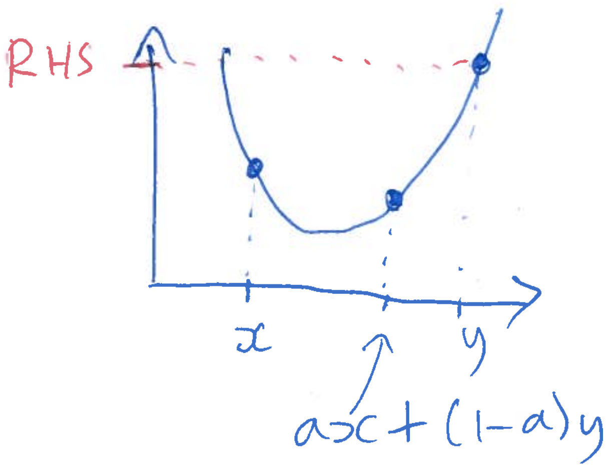
Quasi-convex? ✓

Quasi-concave ✗

Theorem If f is a convex function, then f is a quasi-convex function.

Theorem $f : X \to \mathbb{R}$ is a quasi-convex function if and only if X is convex and for all $x, y \in X$ and $a \in (0,1)$

$$\underbrace{f(ax + (1-a)y)}_{\text{the curve}} \leq \max\{f(x), f(y)\}$$



return from detour:

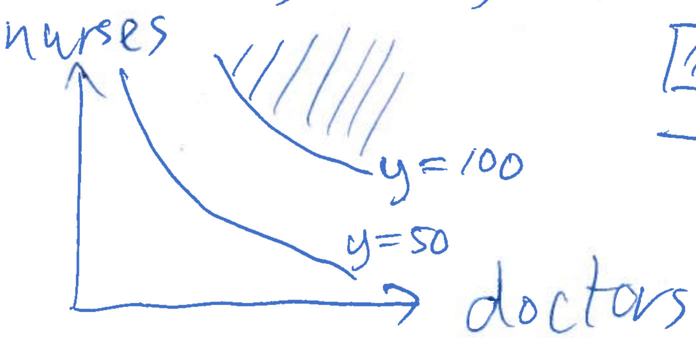
* f is a quasi-concave function.

Rationale: Let y be a target output level. The upper contour set at level y is

$$\{x \in X : f(x) \geq y\},$$

i.e. the set of inputs that meet the production target.

Amounts to assuming that it is possible to "mix" different ways of meeting any production target.



▨ upper contour set

— isoquant (same output quantity along the isoquant)

2.2 Profit maximisation

Only consider perfect competition.

Firms can't influence ~~p~~ prices $p \in \mathbb{R}_+$ (output) and $\underbrace{w}_{\text{e.g. wages (and equipment, etc.)}} \in \mathbb{R}^{N-1}_+$ (input).

The firm's profit function is:

$$\pi(p;w) = \max_{x \in \mathbb{R}^{N-1}_+} \underbrace{p f(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{costs}}$$

$$= p f(\underbrace{x(p;w)}_{\text{factor demand function}}) - w \cdot \underbrace{x(p;w)}$$

First-order condition with respect to x_i:

$$p \frac{\partial f(x^*)}{\partial x_i} = w_i.$$

Divide FOC$-x_i$ by FOC$-x_j$

$$\frac{\frac{\partial f(x^*)}{\partial x_i}}{\frac{\partial f(x^*)}{\partial x_j}} = \frac{w_i}{w_j}.$$

Example: Suppose music recordings are produced from musician labour and technician labour.

Answer:

r royalties (sales price)

l_m musician labours

l_t technician labour

w_m, w_t wages

$f(l_m, l_t)$ song output

$$\pi(r; w_m, w_t) = \max_{l_m, l_t} rf(l_m, l_t) - w_m l_m - w_t l_t.$$

Example Glycerine is a by-product of diesel production, produced from waste organic material.

Answer:

w waste input

$g(w)$ glycerine output

$d(w)$ diesel output

p^w, p^g, p^d prices of waste, glycerine, diesel

$$\pi(p^g, p^d; p^w) = \max_{w} \underbrace{p^g g(w) + p^d d(w)}_{\text{revenue}} - \underbrace{p^w w}_{\text{cost}}$$

Example PET (polyethylene) is made from ethylene, which is made from crude oil. Model a vertically integrated firm that buys oil and sells plastic.

Answer

x crude input

$e = f(x)$ ethylene output

$y = g(e)$ plastic output

p^x, p^y prices of ~~eth~~ crude and plastic

$$\pi(p^y; p^x) = \max_x \; p^y g(f(x)) - p^x x$$

HW: Generalise to buy/sell ethylene.