Banach's fixed point theorem continued...

Existence & convergence

Recall $x_{n+1} = f(x_n)$. Our goal: prove that x_n is a Cauchy sequence.

Recall the contraction property:

$$d(f(x), f(y)) \leq a\, d(x, y).$$

Pick $x = x_0$ and $y = x_1$, we get

$$d(f(x_0), f(x_1)) \leq a\, d(x_0, x_1)$$

$$\Rightarrow d(x_1, x_2) \leq a\, d(x_0, x_1)$$

~~Pick $x = x_n$ and $y = x_{n+1}$ we get~~

$$\cancel{d(x_{n+1}, x_{n+2}) \leq a\, d(x_n, x}$$

Pick $x = x_0$ and $y = x_m$, we get

$$d(f(x_0), f(x_m)) \leq a\, d(x_0, x_m)$$

$$\Rightarrow d(x_1, x_{m+1}) \leq a\, d(x_0, x_m)$$

Also
$$\begin{aligned} d(f^2(x_0), f^2(x_m)) &\leq a\, d(f(x_0), f(x_m)) \\ &\leq a\, a\, d(x_0, x_m) \\ &= a^2 d(x_0, x_m). \end{aligned}$$

$$\Rightarrow d(x_2, x_{m+2}) \leq a^2 d(x_0, x_m).$$

More generally,

$$d(x_n, x_{n+m}) = d(f^n(x_0), f^n(x_m)) \leq a^n \, d(x_0, x_m).$$

We conclude:

$$\boxed{d(x_n, x_{n+m}) \leq a^n \, d(x_0, x_m).}$$

Now, by the triangle inequality:

$$d(x_0, x_m) \leq \underbrace{d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m)}_{\text{very indirect route}}$$

$$\leq d(x_0, x_1) + d(x_1, x_2) + \underbrace{\cdots}_{\text{even larger!}}$$

$$\leq d(x_0, x_1) + a\,d(x_0, x_1) + a^2 d(x_0, x_1) + \cdots + a^n d(x_0, x_1) + \cdots \quad \leftarrow \text{use } \boxed{\cdots}$$

$$\leq d(x_0, x_1)\Big[\underbrace{1 + a + a^2 + a^3 + \cdots}_{\text{geometric series}}\Big]$$

$$= \frac{d(x_0, x_1)}{1 - a}$$

By $\boxed{\cdots}$, we get

$$d(x_n, x_{n+m}) \leq \frac{a^n}{1-a} d(x_0, x_1).$$

So ~~t~~ this implies:

$$d(x_n, x_m) \leq \frac{a^N}{1-a} d(x_0, x_1) \text{ for all } n, m > N.$$

So x_n is a Cauchy sequence.

Since $x_n \in X$ is a Cauchy sequence and X is complete, we conclude $x_n \to x^*$ for some $x^* \in X$. By continuity of f, $f(x_n) \to f(x^*)$. But $f(x_n) = x_{n+1} \to x^*$.
So $f(x_n) \to f(x^*)$ and $f(x_n) \to x^*$.
Therefore $f(x^*) = x^*$. We conclude x^* is a fixed point of f.

<u>Approximation bound:</u>

Since $x_m \to x^*$, and d is continuous,

$$d(x_n, x^*) = \lim_{m\to\infty} d(x_n, x_m)$$

$$\leq \lim_{m\to\infty} \frac{a^n}{1-a} d(x_0, x_1)$$

by formula above

$$= \frac{a^n}{1-a} d(x_0, x_1). \qquad \square$$

In economics, the main application is

$$V(a) = \max_{c,a'} u(c) + \beta \tilde{V}(a')$$
$$\text{s.t. } c + a' = a.$$

$\cancel{V(a)} = f(\tilde{V})(a)$, where f is the "Bellman operator".

If V^* is the actual value function, then $V^* = f(V^*)$. Details $\to$ Appendix G

C9 Compact sets

<u>Det</u> Let A be a subset of (X,d).
We say A is <u>compact</u> if every sequence $x_n \in A$ has a convergent subsequence $y_n \to y^* \in A$.
~~We say a metric~~ If X is compact, we say (X,d) is a <u>compact metric spaces</u>.

<u>Def</u> Any set A in (X,d) is called <u>bounded</u> if $A \subseteq N_r(x^*)$ for some $r > 0$ and some $x^* \in X$.

<u>Theorem (Bolzano-Weierstrass)</u>
Let A be a subset of $(\mathbb{R}^n, d_2)$.
Then A is compact if and only if A is closed and bounded.

<u>Theorem</u> Suppose $f: X \to Y$ is a continuous function between (X, d_X) and (Y, d_Y). If X is compact and $Y = \text{range}(f)$, then Y is compact.

Proof Let $y_n \in Y$ be any sequence. Since $Y = f(X)$, there exists a sequence $x_n \in X$ such that $y_n = f(x_n)$. Since X is compact x_n has a convergent subsequence x_{n_k}. Since f is continuous, $f(x_{n_k})$ is a convergent sequence. But $f(x_{n_k}) = y_{n_k}$ is a subsequence of y_n. So y_n has a convergent subsequence. $\square$

Theorem (Extreme Value Theorem) Suppose $f: X \to \mathbb{R}$ is a continuous function between a non-empty compact metric space (X, d) and $(\mathbb{R}, d_2)$. Then f has a maximum.

Proof Let $Y = f(X)$. (Possible utilities from the menu X.) Since X is compact, (Y, d_2) is compact, and therefore Y is compact inside $(\mathbb{R}, d_2)$. By Bolzano–Weierstrass, Y is closed and bounded. Bounded $\Rightarrow$ $\sup Y$ is finite.

Closed $\Rightarrow \sup Y \in Y$.
So $f^{-1}(\sup Y)$ exists. $\square$