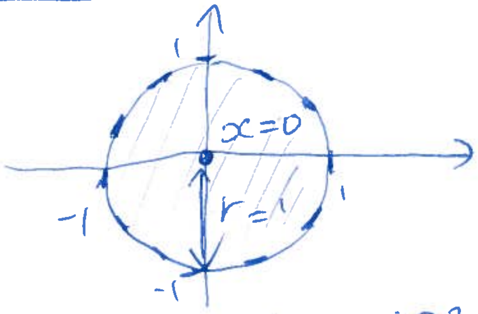
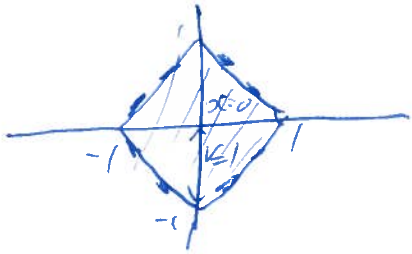
C5 Open sets (cont'd)

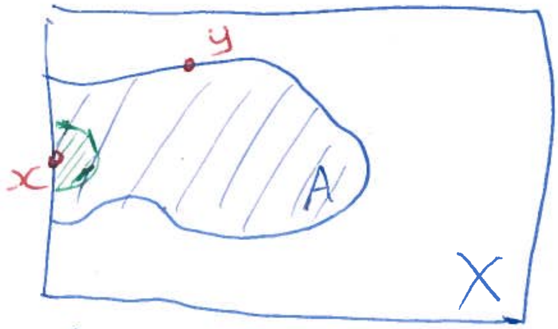


$N_1(0)$ inside $(\mathbb{R}^2, d_2)$



$N_1(0)$ inside $(\mathbb{R}^2, d_1)$

called a ball despite being a square!

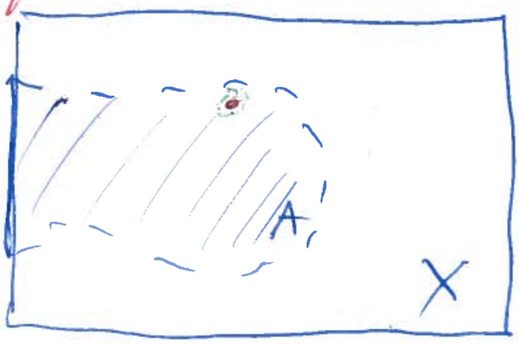


(X, d_2)

$N_r(x)$

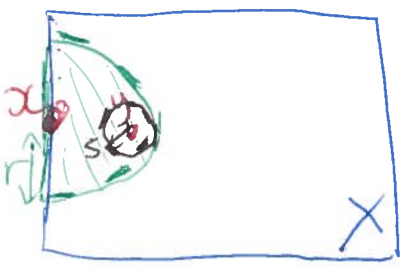
Since $N_r(x) \subseteq A$, x is an interior point of A.

y is not an interior point



A is an open set

<u>claim:</u> Open balls are open sets.



$N_r(x)$

If $s = r - d(x,y)$ then $N_s(y) \subseteq N_r(x)$.

Theorem C5 Let A be a subset of a metric space (X, d). Then A is an open set if and only if A contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

Theorem C6 Let A be any subset of a metric space (X, d). Then A is an open set if and only if $X \setminus A$ is a closed set.

Proof trick: $\partial A = \partial(X \setminus A)$

because if
$$\left.\begin{array}{l} a_n \in A \text{ s.t. } a_n \to x \\ b_n \in X\setminus A \text{ s.t. } b_n \to x \end{array}\right\} \Leftrightarrow x \in \partial A$$

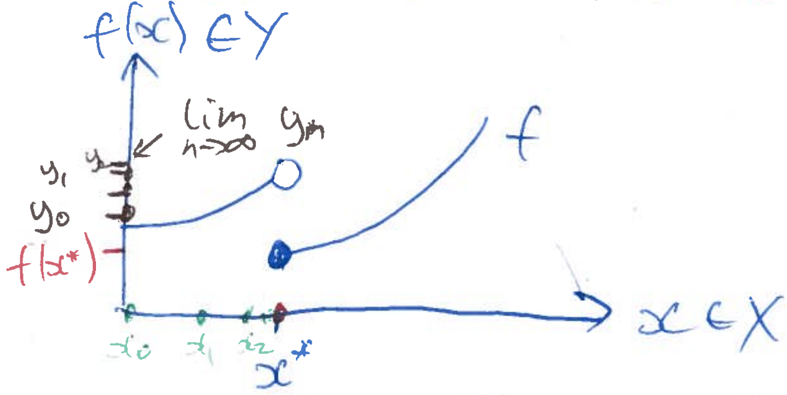
then if
$$\left.\begin{array}{l} \cancel{a_n}\, b_n \in X\setminus A \text{ s.t. } b_n \to x \\ a_n \in (X\setminus(X\setminus A)) = A \\ a_n \to x \end{array}\right\} x \in \partial(X\setminus A)$$

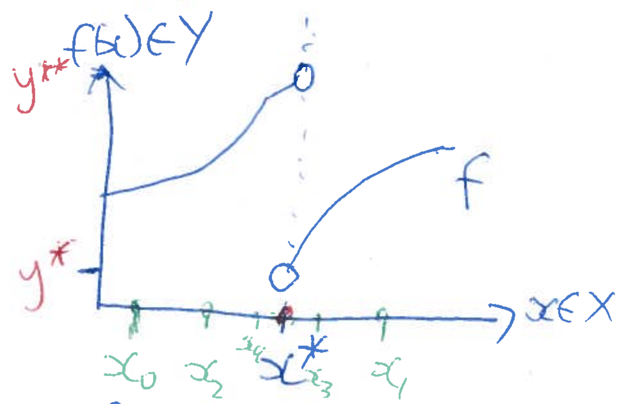
Examples:

* open and closed: $A = [0,1]$ inside $([0,1] \cup [2,3], d_2)$
* " " " : $A = \emptyset$ or $A = X$ in (X, d).
* neither open nor closed: $A = [0,1)$ in $(\mathbb{R}, d_2)$
* open but not closed: $A = [0,1)$ in $([0,\infty), d_2)$
* closed but not open: $A = [0,1]$ in $(\mathbb{R}, d_2)$.

C6 Continuity

Def C11 Consider two metric spaces (X, d_X) and (Y, d_Y). We say that a function $f: X \to Y$ is continuous at $x^* \in X$ if for every sequence $x_n \in X$ converging ~~&~~ with $\underbrace{x_n \to x^*}_{\text{sequence of pre-images}}$, the corresponding sequence $\underbrace{y_n = f(x_n)}_{\text{sequence of images}}$ converges with $f(x_n) \to f(x^*)$. We say f is continuous if it is continuous at all $x^* \in X$.





f is discontinuous at x^*.

② $y_n = f(x_n)$ has infinitely many points near y^* and y^{**}

③ So y_n is non-convergent.

① $x_n \to x^*$

② $y_n = f(x_n)$ has infinitely many points near y^* and y^{**}

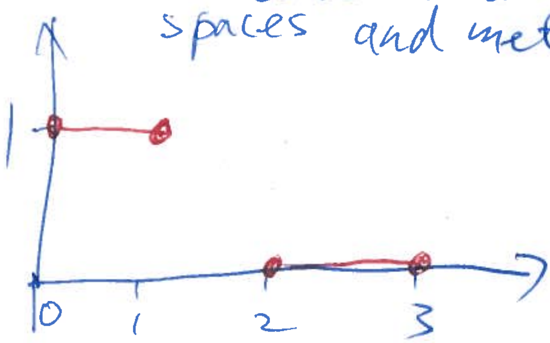
③ So y_n is non-convergent.

Eg: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$

is discontinuous everywhere.

Eg: $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [2,3] \end{cases}$

is continuous with the usual spaces and metrics defined below.



$f: X \to Y$ where $X = [0,1] \cup [2,3]$

$d_X = d_2$

$Y = \mathbb{R}$ and $d_Y = d_2$.

<u>Def</u> $f(A) = \{f(a) : a \in A\}$ <u>images</u>

$f^{-1}(B) = \{x \in X : f(x) \in B\}$ <u>pre-images</u>

<u>inverse image</u>

e.g. $f(x) = x^2$, $f^{-1}([0,1]) = [-1,1]$.

$f: \mathbb{R} \to \mathbb{R}$