4.6 Existence of Equilibrium

Theorem (Existence) Consider a pure-exchange economy in which:

* each $u_h : \mathbb{R}_+^N \to \mathbb{R}$ is continuous, strictly increasing, and strictly quasi-concave, and
* $\sum_h e_{hn} > 0$ for all goods n.

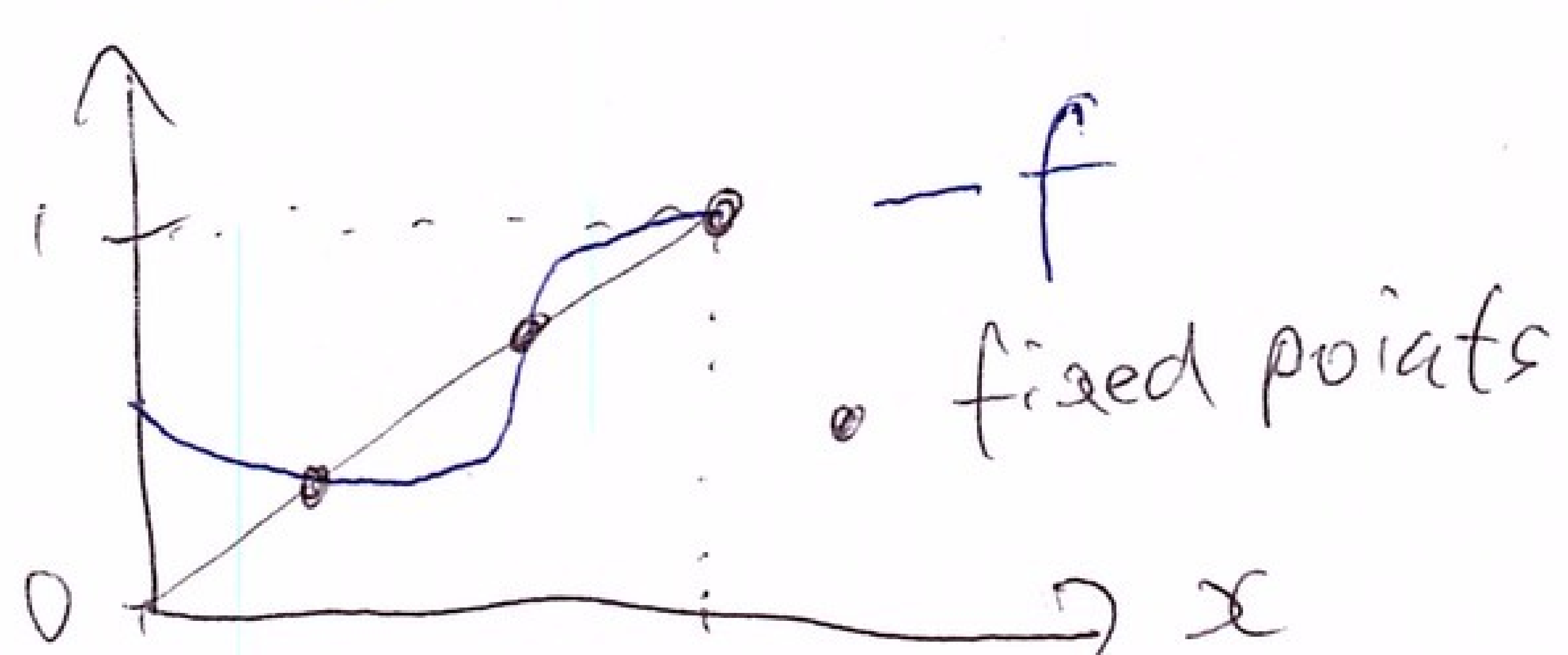
Then there exists a pure-exchange equilibrium (x^*, p^*).

To prove this, we will use a fixed point theorem.

"self-map"

Def Consider a function $f : X \to X$. We say $x^* \in X$ is a <u>fixed point</u> of f if $x^* = f(x^*)$.

Theorem If $f : [0,1] \to [0,1]$ is continuous, f has a fixed point.



Proof Consider $g(x) = f(x) - x$.
We know $g(0) \geq 0$ and $g(1) \leq 0$.
Since g is continuous, it must cross 0 at some point $x^* \in [0,1]$ (by the intermediate value theorem).
Since $g(x^*) = 0 = f(x^*) - x^*$

$$\Rightarrow x^* = f(x^*). \qquad \square$$

Theorem (Brouwer's fixed point theorem)
If $f: X \to X$ is continuous and $X \subset \mathbb{R}^n$ is non-empty, convex, and compact (closed & bounded), then f has a fixed point.

Proof of existence theorem:
Basic strategy: p^* are equilibrium prices $\iff \underbrace{z(p^*)}_{\text{excess demand}} = 0$

assumptions $\Rightarrow z$ is continuous

We will build a "price adjustment" function out of z.

Step 1: "truncated excess demand"

$$\bar{z}_i(p) = \min\{1, z_i(p)\}.$$

Step 2: $p_i' = p_i + \underbrace{\max\{0, \bar{z}_i(p)\}}_{\text{ignore excess supply}}$.

Walras told us: we only "solve" excess demand problems.
Specifically: if $\bar{z}(p) \neq 0$, then there is excess demand in some market i.

Step 3: $p_i'' = \dfrac{p_i'}{\sum_{j=1}^{N} p_j'}$ ← "normalisation" so that they add up to 1.

eg: $p' = (2, 1, 5) \rightarrow$ sum to 8
$p'' = \left(\frac{2}{8}, \frac{1}{8}, \frac{5}{8}\right)$

Prices don't grow without bound.

Finally: $X = \{p \in \mathbb{R}^N_+ : \sum_{n=1}^{N} p_n = 1\}$ is non-empty, convex, compact.
The step-3-adjustment-process does define a continuous function $f: X \to X$.
So Brouwer's fixed point theorem says there is some p^* such that $p^* = f(p^*)$. So $z(p^*) = 0 \Rightarrow p^*$ is an equilibrium price vector. $\square$

4.7 Implementation of Efficient Allocations

Def (Pure exchange equilibrium with lump-sum taxes)

Consider a pure exchange economy with N goods and $(u_h)_{h\in H}$ utility functions and $(e_h)_{h\in H}$ endowments. Given taxes $(t_h)_{h\in H}$ such that $\sum_{h\in H} t_h = 0$ ("budget balanced"), we say that (x^*, p^*) is an equilibrium if

(i) $$x_h^* \in \underset{x_h \in \mathbb{R}_+^N}{\arg\max}\; u(x_h) \quad \text{s.t. } p\cdot x_h \leq p\cdot e_h - t_h$$

and

(ii) $$\sum_{h\in H} x_h^* = \sum_{h\in H} e_h^*.$$

Theorem (Second Welfare Theorem)

Consider a pure exchange economy in which each u_h is continuous, increasing, and strictly quasi-concave and $\sum_h e_{hn} > 0$ for all n.

If $\underbrace{x^* \in \mathbb{R}^{NH}_{++}}_{\text{a little bit of everything to everyone}}$ is Pareto efficient then there exists a budget-balanced lump-sum tax policy $(t^*_h)_{h \in H}$ such that there exists prices $p^* \in \mathbb{R}^N_+$ for which (x^*, p^*, t^*) is a pure exchange equilibrium with lump-sum transfers.

Proof

Step 1 Figure out prices.

Consider the pure exchange economy with endowments $e_h = x^*_h$ for all $h \in H$.

By the existence theorem, there are prices p^* such that (x^{**}, p^*) is an equilibrium. Claim (x^*, p^*) is also an equilibrium. Why? Since x^* is efficient and x^{**} has to be weakly better for all households $\Rightarrow u_h(x^{**}_h) = u_h(x^*_h)$ for all h.

Step 2: taxes

~~Proo~~ Household h starts with

$$p^* \cdot e_h \quad \leftarrow \text{money, not goods}$$

worth of goods. The goal is that they consume $p^* \cdot x_h^*$ worth of goods.

Set $t_h^* = -[p^* \cdot x_h^* - p^* \cdot e_h]$.

Note: $\sum_{h \in H} t_h^* = 0$ because $\sum_h x_h^* = \sum_h e_h$.

Step 3: check that we have an eq.

Note that the budget constraints in step 1 & 2 are the same.

So since (x^*, p^*) was an equilibrium when endowments are x^*,

(x^*, p^*) is an equilibrium with endowments e and taxes t^*. $\square$

* define notation
* choice variables under max/min.
* does ~~be~~ every choice a cost and a benefit. [are exceptions]
* don't forget dividends
* profit function with prices as state variable.
* market clearing:

N = # markets = # market clearing conditions

= # prices

6 (iii) Each input choice ~~…~~ (L_0, H_0), there is a function of prices

$$g(p, r_0, w; L_0, H_0) = pf(L_0, H_0) - r_0 L_0 - w H_0$$

The profit function is the upper envelope of these functions,

$$\pi^0(p, r_0, w) = \max_{L_0, H_0} g(p, r_0, w; L_0, H_0).$$

Since $g(\cdot, \cdot, \cdot; L_0, H_0)$ is linear, it is a convex function. Moreover, the upper envelope of convex functions is convex. We conclude π^0 is convex.