3.6 Slutsky decomposition

$$h(p, u) = x(p, e(p, u)). \quad (*)$$

Theorem (Slutsky equation) If $u : \mathbb{R}^N_+ \to \mathbb{R}$ is smooth and the policy functions $x(p, m)$ and $h(p, u)$ are differentiable, then

$$\underbrace{\frac{\partial x_i(p, m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p, u)}{\partial p_j}\right]_{u = v(p, m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p, m)}_{\substack{\text{wealth} \\ \text{lost}}} \frac{\partial x_i(p, m)}{\partial m}}_{\text{income effect}}$$

eg. j = food

i = ~~fanciness~~ ~~lot listed~~ days

Proof Rewrite $(*)$:

$$h_i(p, u) = x_i(p, e(p, u)).$$

Diff both sides w.r.t. p_j:

a marginal value

$$\frac{\partial h_i(p, u)}{\partial p_j} = \left[\frac{\partial x_i(p, m)}{\partial p_j} + \frac{\partial x_i(p, m)}{\partial m} \frac{\partial e(p, u)}{\partial p_j}\right]_{m = e(p, u)}$$

envelope theorem

$$= \left[\frac{\partial x_i(p, m)}{\partial p_j} + \frac{\partial x_i(p, m)}{\partial m} h_j(p, u)\right]_{m = e(p, u)}$$

We can now re-arrange to get the Slutsky equation.

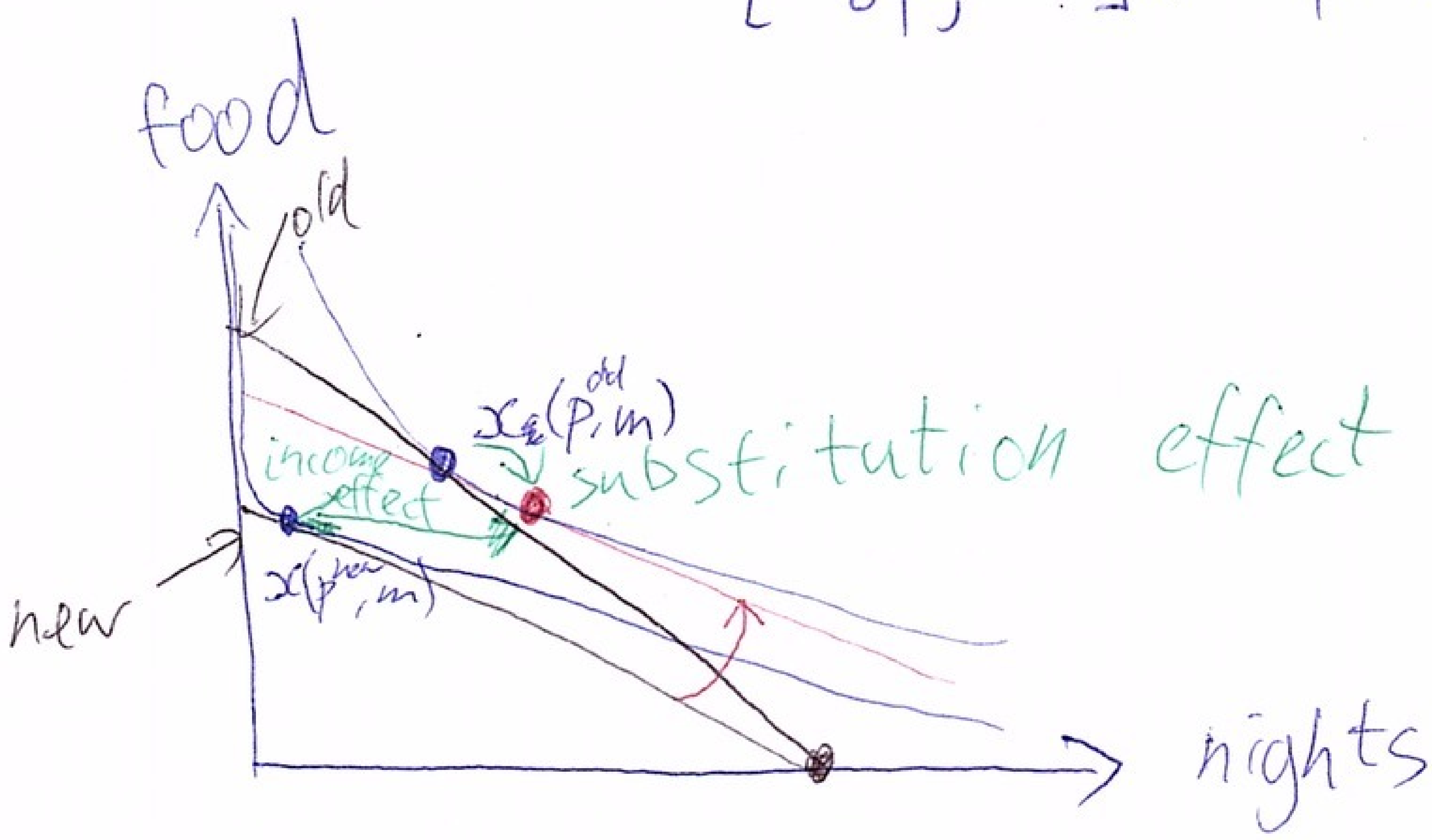
$$\left[\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} h_j(p,u)\right]_{m=e(p,u)} = \frac{\partial h_i(p,u)}{\partial p_j}$$

$$\left[\frac{\partial x_i(p,m)}{\partial p_j}\right]_{m=e(p,u)} = \frac{\partial h_i(p,u)}{\partial p_j} - \left[h_j(p,u)\frac{\partial x_i(p,m)}{\partial m}\right]_{m=e(p,u)}$$

$$\frac{\partial x_i(p,m)}{\partial p_j} = \left[\frac{\partial h_i(p,u)}{\partial p_j}\right]_{u=v(p,m)} - \left[h_j(p,u)\frac{\partial x_i(p,m)}{\partial m}\right]_{u=v(p,m)}$$

$$= \left[\frac{\partial h_i(p,u)}{\partial p_j}\right]_{u=v(p,m)} - x_j(p,m)\frac{\partial x_i(p,m)}{\partial m}$$

$\square$



Chapter 4 Equilibrium

4.1 Economies (or environment)

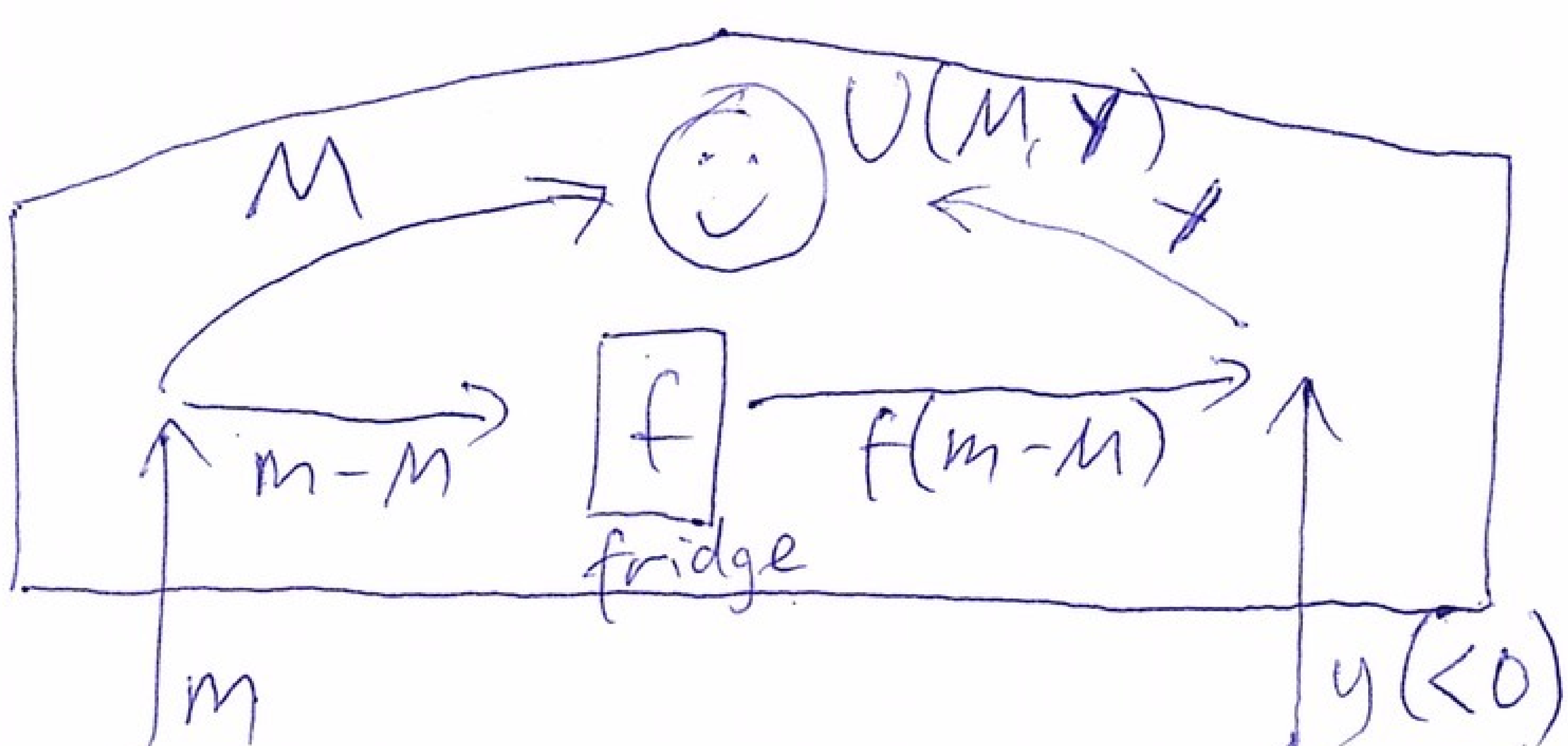
Def Pure exchange economy with N goods and H households consists of:

* a utility function $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ for each household $h \in H$, and
* an endowment $e_h \in \mathbb{R}^N_+$ for each household $h \in H$.

Def An allocation specifies consumption $x_h \in \mathbb{R}^N_+$ for each household $h \in H$.

Def An allocation x is feasible if $\sum_{h \in H} x_h = \sum_{h \in H} e_h$, i.e. for each good g,

$$\sum_{h \in H} x_{hg} = \sum_{h \in H} e_{hg}.$$



home production (m, y): market sees

(M, Y): consumed

$$u(m, y) = \max_{M, Y} U(M, Y) \leftarrow \text{"real utility"}$$

$$\text{s.t. } \underbrace{Y}_{\text{yoghurt consumed}} = \underbrace{f(m - M)}_{\text{yoghurt produced}} - \underbrace{(-y)}_{\text{yoghurt sold}}$$

<0

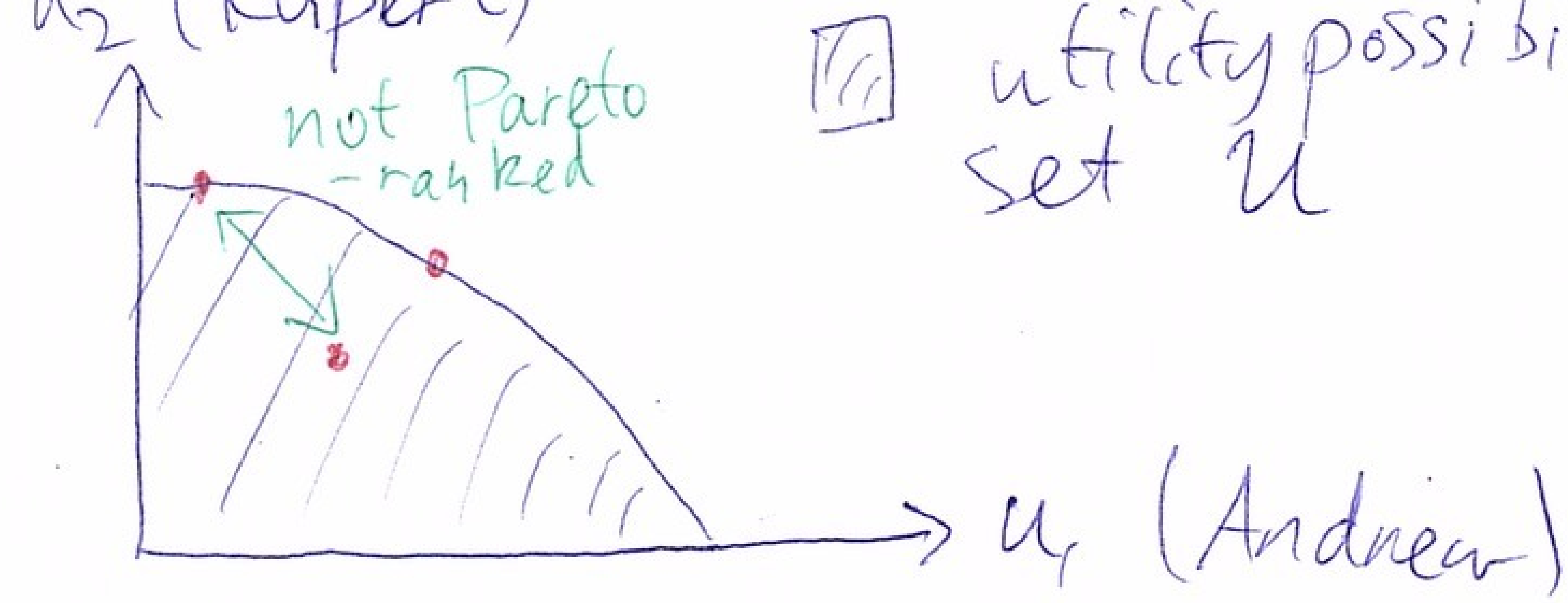
4.2 Efficient Allocations

Def The <u>utility possibility set</u> of an economy is the set of feasible vectors of utilities for each household. Specifically, in a pure exchange economy, this is

$$\mathcal{U} = \{ (u_h(x_h))_{h \in H} : x_h \in \mathbb{R}^N_+ \text{ for all } h \in H \text{ and } \sum_{h \in H} x_h = \sum_{h \in H} e_h \}$$

Trick: free disposal:

$$\sum_{h \in H} x_{hg} \le \sum_{h \in H} e_{hg}$$



▨ utility possibility set $\mathcal{U}$

<u>Def</u> A vector utilities $u \in \mathbb{R}^H$ <u>Pareto dominates</u> another vector $u' \in \mathbb{R}^H$ if:

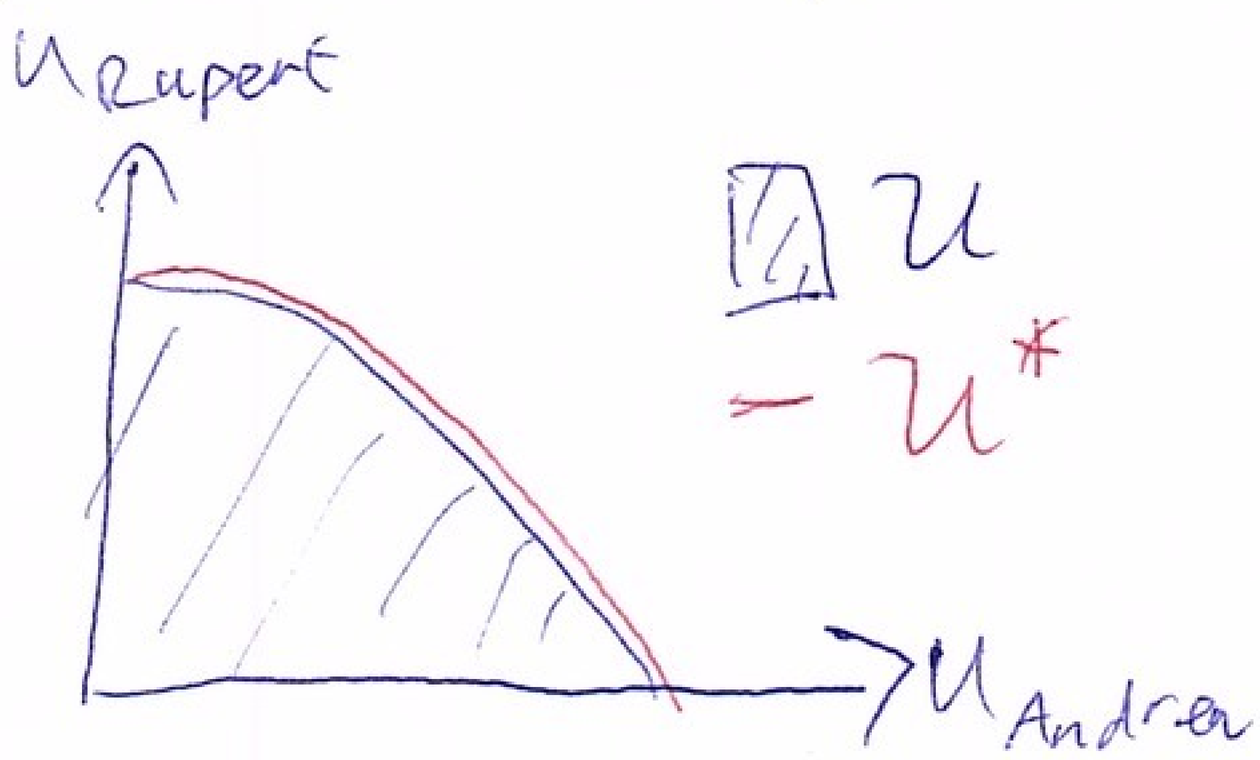
(i) no household is worse of, i.e. $u_h \geq u'_h$ for all $h \in H$, and

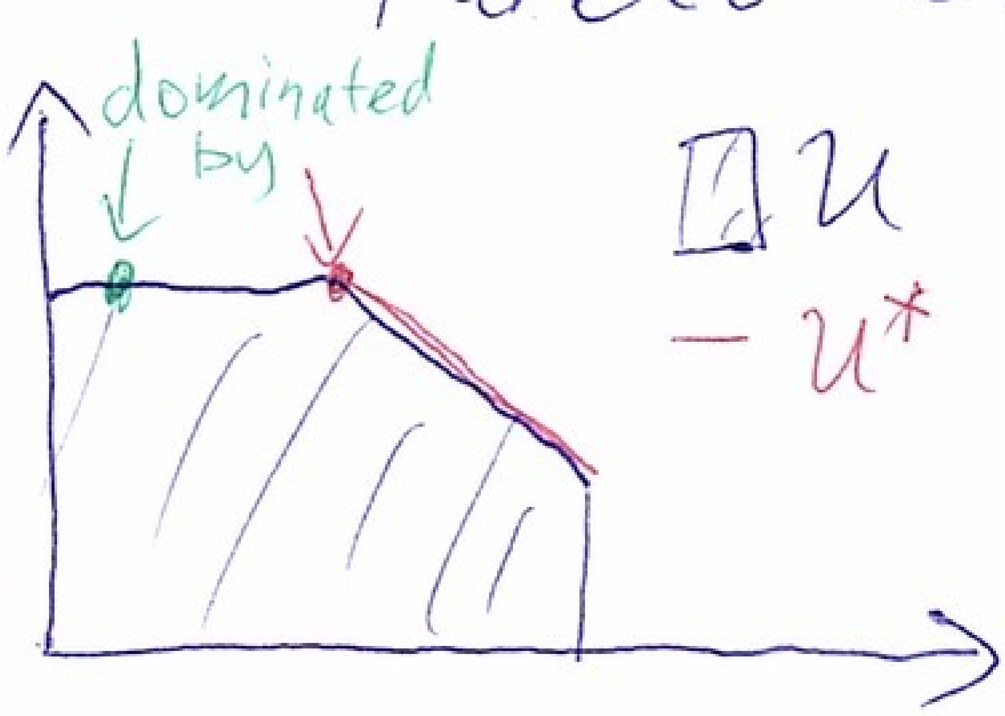
(ii) (at least) one household is strictly ~~prefers~~ better off. i.e. $u_{h^*} > u'_{h^*}$ for some $h^* \in H$.

<u>Def</u> If u Pareto dominates u', then we say that u' is <u>Pareto inefficient</u>.

<u>Def</u> If u is not Pareto inefficient, (i.e. is not dominated by any $u' \in \mathbb{R}^H$), then we say u is <u>Pareto efficient</u>.

<u>Def</u> The <u>Pareto frontier</u> of a utility possibility set $\mathcal{U}^* = \{u \in \mathcal{U} : u \text{ is Pareto efficient}\}$.





Def A *social welfare function* is any function $W : \mathbb{R}^H \to \mathbb{R}$.

Theorem Let $\mathcal{U} \subseteq \mathbb{R}^H$ and $W : \mathbb{R}^H \to \mathbb{R}$.
If $u \in \mathcal{U}$ maximises welfare, i.e.

$$u \in \underset{\hat{u} \in \mathcal{U}}{\arg\max}\, W(\hat{u})$$

and W is strictly increasing, then u is Pareto efficient, i.e. $u \in \mathcal{U}^*$.

Proof Suppose for the sake of contradiction that $\hat{u} \in \mathcal{U}$ and $\hat{u}$ dominates u. But since W is increasing, this would imply $W(\hat{u}) > W(u)$ ↯

4.3 Equilibrium

Def Consider a pure-exchange economy with utility functions $(u_h)_{h \in H}$ and endowments $(e_h)_{h \in H}$. We say that (x^*, p^*) consisting of an allocation $x^* \in \mathbb{R}_+^{HN}$ and prices $p^* \in \mathbb{R}^N$ form a pure-exchange equilibrium if:

(i) for each household $h \in H$,

$$x_h^* \in \arg\max_{x_h \in \mathbb{R}_+^N} u_h(x_h)$$
$$\text{s.t. } p^* \cdot x_h \leq p^* \cdot e_h,$$

and

(ii) $\sum_h x_h^* = \sum_h e_h.$

Question 3. (Micro 1 class exam in December 2012) A farm produces food from labour. However, the farm does not have a distribution network, so it can not sell the food directly to the households. Rather, it must sell the food to a supermarket at a wholesale price, which then resells to households at a retail price. The supermarket buys food and labour, which it uses to resell the food. Some food might get wasted; more labour means less food gets wasted. All households are identical, and supply labour to both firms.

(i) Formulate an economy by writing down the households' and firms' value functions, and the market clearing conditions. Focus attention on symmetric equilibria, i.e. in which all households make the same decisions. (Hint: you might find it helpful to consider the wholesale food a completely separate good. Don't forget profits.)

Answer: **Household.** p retail food price, w wage, c consumption, l labour, H number of households, $u(c, l)$ utility function, $\Pi = \Pi^F + \Pi^S$ firms' profits, value

different

* check: every choice has cost & a benefit

$$v(p, w) = \max_{c,l} u(c, l)$$

prices choices

$$\text{s.t. } pc = wl + \frac{\Pi}{H}$$

dividends

Farm. D_F wholesale good produced, $D_F = f(L_F)$ production function, ϕ wholesale price, value

LF farm's labour input

N = #types of goods = #prices = #market clearing conditions

$$\pi^F(\phi, w) = \max_{L_F} \phi f(L_F) - wL_F.$$

Supermarket. D_S wholesale good purchased, C_S retail food sold, $C_S = g(L_S, D_S)$ production function, value

$$\pi^S(p, \phi, w) = \max_{L_S, D_S} pg(L_S, D_S) - \phi D_S - wL_S.$$

* every market should buyers & sellers

Equilibrium. A symmetric allocation consists of quantities for households (c^*, l^*), the farm (D_F^*, L_F^*), and the supermarket (C_S^*, D_S^*, L_S^*). These choices, along with prices (p^*, ϕ^*, w^*) and profits (Π^{F*}, Π^{S*}) form an equilibrium if the

- choices solve the problems defined above,
- profits match: $\Pi^{S*} = \pi^S(p^*, \phi^*, w^*)$ and $\Pi^{F^*} = \pi^F(\phi^*, w^*)$.
- food clears: $Hc^* = C_S^*$.

(*) feasible allocations

Question 3 - redoing with separate labour markets

l_S, l_F ~~labour of~~ retail labour supply,
farm " "

w_S, w_F wages for retail/farm labour

$$\max_{c, l_S, l_F} u(c, l_S, l_F) \leftarrow \text{could be } u(c, l_S + l_F)$$

$$\text{s.t. } pc = w_S l_S + w_F l_F + \frac{\pi}{H}$$

Farm: $\pi^F(\phi, w_F) = \max_{L_F} \phi f(L_F) - w_F L_F.$

Supermarket: $\pi^S(p, \phi, w_S) = \max_{L_S, D_S} p\, g(L_S, D_S) - w_S L_S.$

market clearing:

$p: \quad Hc = g(L_S, D_S)$

$w_S: \quad Hl_S = L_S$

$w_F: \quad Hl_F = L_F$

$\phi: \quad D_S = f(L_F)$

4.4 Characterising Equilibria

Def The <u>excess demand function</u> is

$$z(p) = \sum_{h \in H} (\underbrace{x_h(p)}_{\text{household demand function}} - e_h).$$

Note: p is an equilibrium vector

$$\Leftrightarrow z(p) = (\underbrace{0, \ldots, 0}_{N \text{ zeros}}) = 0$$

Theorem (Walras' law) Consider a pure exchange economy $(u_h, e_h)_{h \in H}$, with strictly increasing utility functions.

(i) $p \cdot z(p) = 0$ for all $p \in \mathbb{R}^N_{++}$.

not just equilibrium

(ii) If $N-1$ markets clear at prices $p \in \mathbb{R}^N_{++}$, then all markets clear ($\Rightarrow p$ is an equilibrium price vector).

(iii) For every $p \in \mathbb{R}^N_{++}$, the market does not clear (i.e. $z(p) \neq 0$)

$\Leftrightarrow$ (a) there is a market i s.t. $z_i(p) > 0$.
i.e. there is excess demand in market i, AND

(b) there is a market j s.t. $z_j(p) < 0$, i.e. there is excess supply in market j.

Proof

(i) Since u_h is increasing,

$$p \cdot x_h(p) = p \cdot e_h$$

for all $h \in H$.

Summing up:

$$\sum_{h \in H} p \cdot x_h(p) = \sum_{h \in H} p \cdot e_h$$

$$\Leftrightarrow \sum_{h \in H} p \cdot [x_h(p) - e_h] = 0$$

$$\Leftrightarrow p \cdot \sum_{h \in H} [x_h(p) - e_h] = 0$$

$$\Leftrightarrow p \cdot \quad z(p) \quad = 0.$$

(ii) Without loss of generality, assume that ~~the~~ markets $1, 2, \ldots, N-1$ all clear, i.e. $z_1(p) = 0$, $z_2(p) = 0$, $\ldots$, $z_{N-1}(p) = 0$.

~~Suppose for the sake of contradiction that $z_N(p) \neq 0$.~~

Summing up: $\displaystyle\sum_{j=1}^{N-1} p_j \underbrace{z_j(p)}_{\text{assumed} = 0} = 0$

Subtracting ~~it~~ this from (i):

$$p_N z_N(p) = 0 \Leftrightarrow z_N(p) = 0.$$

(iii) If $z(p) \neq 0$, we must show for some i, j, $z_i(p) > 0$ and $z_j(p) < 0$.

Without loss of generality, suppose $z_i(p) > 0$. We need to prove there is some $z_j(p) < 0$. If this were not the case, i.e. all $z_1(p) \geq 0$, $z_2(p) \geq 0$, $\ldots$, $z_N(p) \geq 0$,

then $p \cdot z(p) > 0$. Contradicts (i). $\lightning$