2.5 Upper Envelopes with Constraint

General view:

$$V(a) = \max_b \; v(a, b) \qquad = v(a, b(a))$$

$$\text{s.t.} \quad \underbrace{w(a, b) \geq 0}_{\text{constraint}}$$

Usual solution: via Lagrangian

$$L(a, b, \lambda) = v(a, b) + \lambda w(a, b).$$

FOC: $\left.\frac{\partial L(a,b,\lambda)}{\partial b}\right|_{b=b(a)} = 0.$

$$\left[v_b(a, b) + \lambda w_b(a, b)\right]_{\substack{b=b(a)\\ \lambda=\lambda(a)}} = 0.$$

__Theorem (Envelope)__ If V, v, w, b, λ are differentiable at $(a, b(a))$, and if the constraint binds at a, i.e., $w(a, b(a)) = 0$, then

$$V'(a) = \left[\frac{\partial v(a,b)}{\partial a} + \lambda \frac{\partial w(a,b)}{\partial a}\right]_{\substack{b=b(a)\\ \lambda=\lambda(a)}}$$

Proof chain rule approach.

$$V(a) = v(a, b(a)).$$

Rewrite: $$V(a) = v(a, b(a)) + \lambda(a) \overbrace{w(a, b(a))}^{=0 \text{ (binds)}}$$
$$= L(a, b(a), \lambda(a)).$$

Differentiate both sides:

$$V'(a) = \left[\frac{\partial L}{\partial a} + \underbrace{\frac{\partial L}{\partial b}}_{\substack{=0 \\ \text{by FOC}}} b'(a) + \underbrace{\frac{\partial L}{\partial \lambda}}_{\substack{=0 \\ = w(a,\, b(a)) \text{ binds}}} \lambda'(a) \right]_{\substack{b = b(a) \\ \lambda = \lambda(a)}}$$

$$= \left[\frac{\partial L(a, b, \lambda)}{\partial a} \right]_{\substack{b = b(a) \\ \lambda = \lambda(a)}}$$

$$= v_a(a, b(a)) + \lambda w_a(a, b(a)). \qquad \square$$

$$= \left[v_a(a, b) + \lambda w_a(a, b) \right]_{\substack{b = b(a) \\ \lambda = \lambda(a)}}$$

Apply to the cost function, whose Lagrangian is

$$L(y, w; x; \lambda) = w \cdot x - \lambda [f(x) - y].$$

FOC x_i: $$w_i = \lambda(y, w) \left. \frac{\partial f(x)}{\partial x_i} \right|_{x = x(y, w)}.$$

envelope theorem:

to y: $$\frac{\partial c(y, w)}{\partial y} = [0 + \lambda]_{\substack{x = x(y, w) \\ \lambda = \lambda(y, w)}} = \lambda(y, w).$$

Apply envelope theorem to w_i:

$$\frac{\partial c(y,w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\left(w\cdot x - \lambda(f(x)-y)\right)\right]_{\substack{x=x(y,w)\\ \lambda=\lambda(y,w)}}$$

$$= [x_i]_{x=x(y,w)}$$

$$= x_i(y,w).$$

<u>Theorem</u> If v is convex and w is quasi-concave, then

$$V(a) = \min_b v(a,b) \quad \text{s.t. } w(a,b) \geq 0$$

is convex. [There is a similar theorem for max problems and v concave]

Apply to cost function

<u>Theorem</u> If the production function f is concave, then the cost function is convex in output, i.e. $c(\cdot, w)$ is ~~ ~~ convex for all w.

<u>Proof</u>

The objective ~~ ~~ $v(x,y) = \overbrace{w\cdot x}^{\text{not concave in } (w,x).}$ is linear, and hence convex in (x,y).

The constraint $w(x,y) = f(x) - y$ is concave.

By the theorem, $c(\cdot, w)$ is convex. □

Chapter 3 Consumption

3.1 Utility functions

As before, N goods, possible consumption levels $\mathbb{R}^N_+$.

Def Consider two possible choices $x, y \in \mathbb{R}^N_+$. If a consumer weakly prefers x over y, we write $x \succsim y$.
If it is a strict preference, we write $x \succ y$, shorthand for $x \succsim y$ but $y \not\succsim x$.
If $x \succsim y$ and $y \succsim x$, then the consumer is indifferent between x and y, written $x \sim y$.

Def A utility function is a function $u : \mathbb{R}^N_+ \to \mathbb{R}$. We say that u represents the preferences $\succsim$ if they agree, i.e. $u(x) \geq u(y) \Leftrightarrow x \succsim y$.

Possible assumptions:

* complete: for all $x, y \in \mathbb{R}^N_+$, either $x \succsim y$ or $y \succsim x$.
  or both (that's how we use the word or)

* reflexive: for all $x \in \mathbb{R}^N_+$, $x \succsim x$ (or equivalently, $x \sim x$).

* transitive: for all $x, y, z \in \mathbb{R}^N_+$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

* continuity: all upper contour sets are closed, $U(x) = \{y \in \mathbb{R}^N_+ : y \succsim x\}$

Theorem * Consider a preference relation $\succsim$. There exists some continuous utility function that represents $\succsim$ iff $\succsim$ is complete, reflexive, transitive and continuous.

* convex: all upper contour sets are convex. $U(x) = \{y \in \mathbb{R}^N_+ : y \succsim x\}$

3.2 Time Preference

$t \in \{1, \dots, T\}$ time

N goods

$\Rightarrow NT$ choices to be made

$J \subseteq \{1, \dots, T\}$

$-J = \{1, \dots, T\} \setminus J$

If $x \in X^T$ then I write

$$x_J = (x_t)_{t \in J}$$

e.g. $N=1$, $T=3$, $x = (1,2,3)$

$y = (4,5,6)$

$J = \{2,3\}$ $x_J = (2,3)$, $y_J = (5,6)$

Notation: $(x_J, y_{-J}) = (\underbrace{4}_{-J \Rightarrow y}, \underbrace{2, 3}_{J \Rightarrow x})$

<u>Def</u> Suppose there are N goods and T periods, and let $X = \mathbb{R}^N_+$. We say $\succsim$ are <u>time separable</u> if for any pair choices $x, y \in X^T$ that coincide in periods J (i.e. $x_J = y_J$) then the preferences are unaffected by simultaneous changes to J-period choices i.e. if $x \succsim y$ then $(z_J, x_{-J}) \succsim (z_J, y_{-J})$

<u>Example</u> $N=1$ (methamphetamine)

0 or 1 per day, $T=4$ days

$2^4=16$ possible choices.

$x = 0000$

$y = 0011$

$x' = 0100$

$y' = 0111$

<u>Andrew:</u> $x \succsim y$

$y' \succsim x'$ (avoid "cold turkey")

Are Andrew's preferences ~~on~~ time-separable?

<u>No:</u> $J=\{2\}$, $z=(1,1,1,1)$.

$x'=(z_J, x_{-J})$, $y'=(z_J, y_{-J})$

$x \succsim y \not\Rightarrow x' \succsim y'$ ↯

<u>Theorem (* Debreu 1960)</u> If $\succsim$ over choices in X^T are complete, reflexive, transitive, continuous, time-separable, strictly increasing and $T \geq 3$, then there exist continuous utility functions $u_1, \ldots, u_T : X \to \mathbb{R}$ such that

$$U(x) = \underbrace{u_1(x_1)}_{\text{day 1's choices}} + u_2(x_2) + \cdots + u_T(x_T)$$

additively separable utility

represents $\succsim$.

Discounted utility:

$$U(x) = u(x_1) + \beta u(x_2) + \cdots + \beta^{T-1} u(x_T).$$

Example cake of size k_1.

$$V_t(k_t) = \max_{x_t, \dots, x_T \geq 0} u_t(x_t) + \cdots + u_T(x_T)$$
$$\text{s.t. } x_t + \cdots + x_T = k_t.$$

Bellman equation:

$$V_t(k_t) = \begin{cases} u_T(k_T) & \text{if } t = T \\ \displaystyle\max_{x_t, k_{t+1} \geq 0} u_t(x_t) + V_{t+1}(k_{t+1}) & \text{if } t < T \\ \text{s.t. } x_t + k_{t+1} = k_t \end{cases}$$

3.3 Utility Maximisation

$$v(p, \underset{\substack{\uparrow \\ \text{money}}}{m}) = \max_{x \in \mathbb{R}^N_+} u(x) = u(\underbrace{x(p, m)}_{\text{demand function}})$$
$$\text{s.t. } p \cdot x \leq m$$

indirect utility function (a value function)

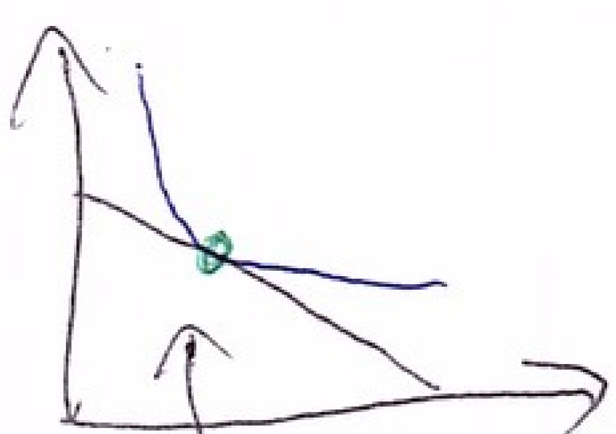
$$v^*(p, \underset{\substack{\uparrow \\ \text{endowment} \\ \in \mathbb{R}^N_+}}{e}) = \max_{x \in \mathbb{R}^N_+} u(x)$$
$$\text{s.t. } p \cdot x \leq p \cdot e$$
$$\iff p \cdot (x - e) \leq 0$$

$$\underline{\text{FOC's } x_i}: \quad \left[\frac{\partial u(x)}{\partial x_i} - \lambda p_i\right]_{\substack{x = x(p,m) \\ \lambda = \lambda(p,m)}} = 0$$

$$\Longleftrightarrow \quad \frac{\left.\frac{\partial u(x)}{\partial x_i}\right|_{x=x(p,m)}}{p_i} = \lambda(p,m)$$

$$\frac{\text{FOC } x_i}{\text{FOC } x_j}: \quad \left| - \frac{\frac{\partial u(x)}{\partial x_i}}{\frac{\partial u(x)}{\partial x_j}} \right| = \frac{p_i}{p_j}$$

slope of indiff curve (implicit function theorem)



same slope at optimal choice

3.4 Consumer's value & policy functions

Envelope theorem:

$$\frac{\partial v(p,m)}{\partial m} = \lambda(p,m)$$

$$\frac{\partial v(p,m)}{\partial p_i} = -\lambda(p,m)\, x_i(p,m) = -\frac{\partial v(p,m)}{\partial m} x_i(p,m)$$

rearrange: $x_i(p,m) = -\dfrac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}$.

what a mess!

* normal good: $\frac{\partial x_i(p,m)}{\partial m} > 0$

* inferior good: $\frac{\partial x_i(p,m)}{\partial m} < 0$

* Giffen good: $\frac{\partial x_i(p,m)}{\partial p_i} > 0$

e.g. bread (for poor families)

* substitutes: i and j are substitutes if $\frac{\partial x_i(p,m)}{\partial p_j} > 0$.

$\frac{\partial x_i(p,m)}{\partial p_j} = \frac{\partial x_j(p,m)}{\partial p_i}$ (later)

* complements: i and j are complements if $\frac{\partial x_i(p,m)}{\partial p_j} < 0$.

3.5 Expenditure & policy functions

$$e(p,\bar{u}) = \min_{x \in \mathbb{R}^N_+} p \cdot x \quad \text{s.t. } u(x) \geq \bar{u} \quad = p \cdot \underbrace{h(p,\bar{u})}_{\text{Hicksian demand}}$$

$\bar{u}$: utility target

expenditure function

Bellman eq: $v(p,m) = \max_u u$ s.t. $e(p,u) = m$.

Applying the envelope theorem to the expenditure function:

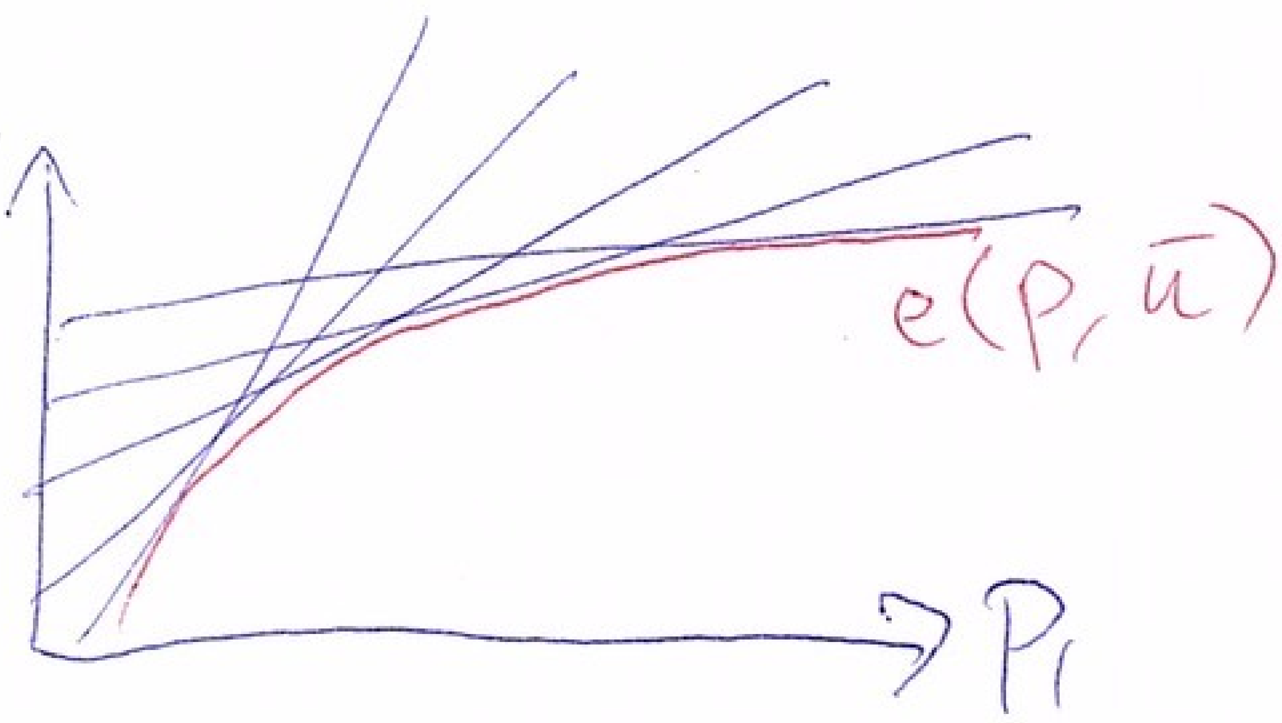
$$\frac{\partial e(p, \bar{u})}{\partial p_i} = h_i(p, \bar{u}).$$

$$= \left[\frac{\partial}{\partial p_i}\left(p \cdot x - \mu[u(x) - \bar{u}]\right)\right]_{x=h(p,\bar{u})}$$

$$= [x_i]_{x=h(p,\bar{u})}$$

$$= h_i(p, \bar{u}).$$

$$\frac{\partial e(p, \bar{u})}{\partial \bar{u}} = \mu(p, u)$$



By the same theorem as before (2.2), e is concave in prices.

$$\Rightarrow \frac{\partial^2 e(p, \bar{u})}{\partial p_i^2} = \underbrace{\frac{\partial h_i(p, \bar{u})}{\partial p_i}}_{\text{substitution effect}} < 0.$$