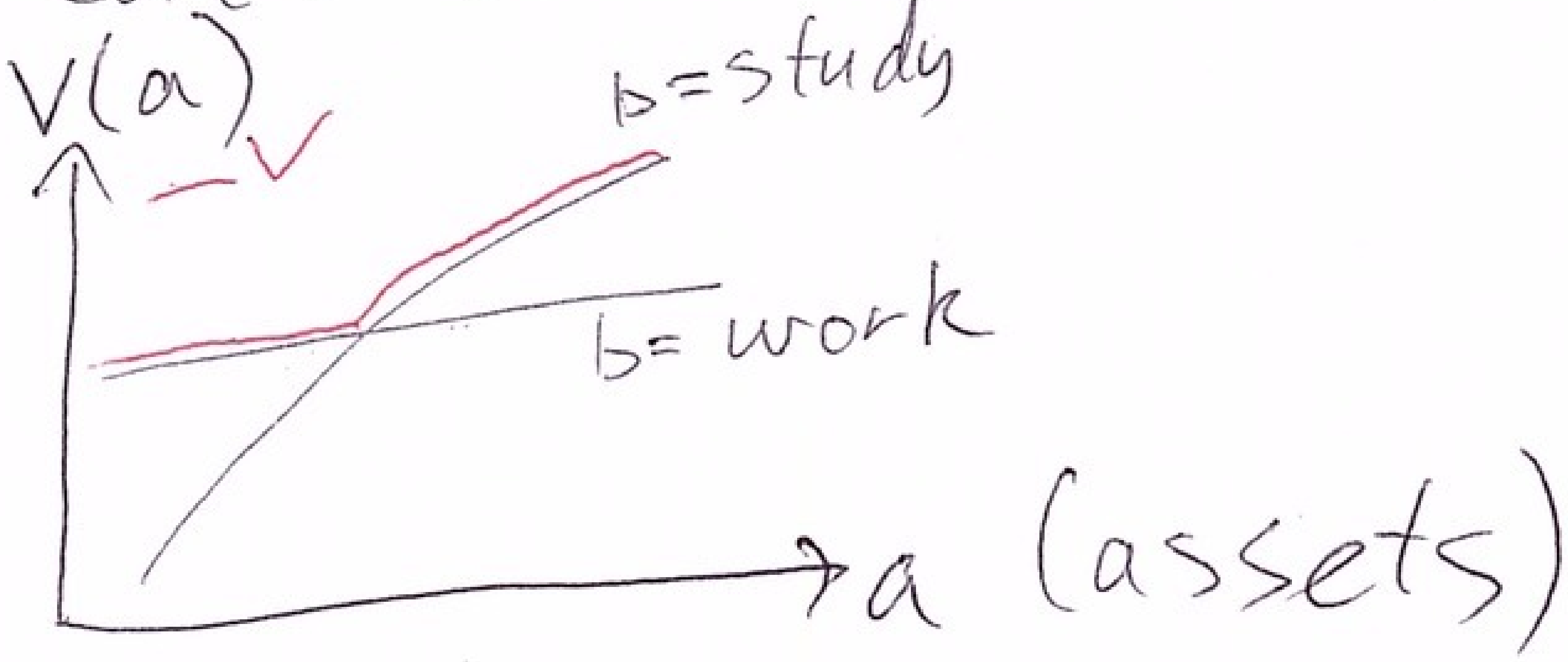
## 2.3 Envelope theorem

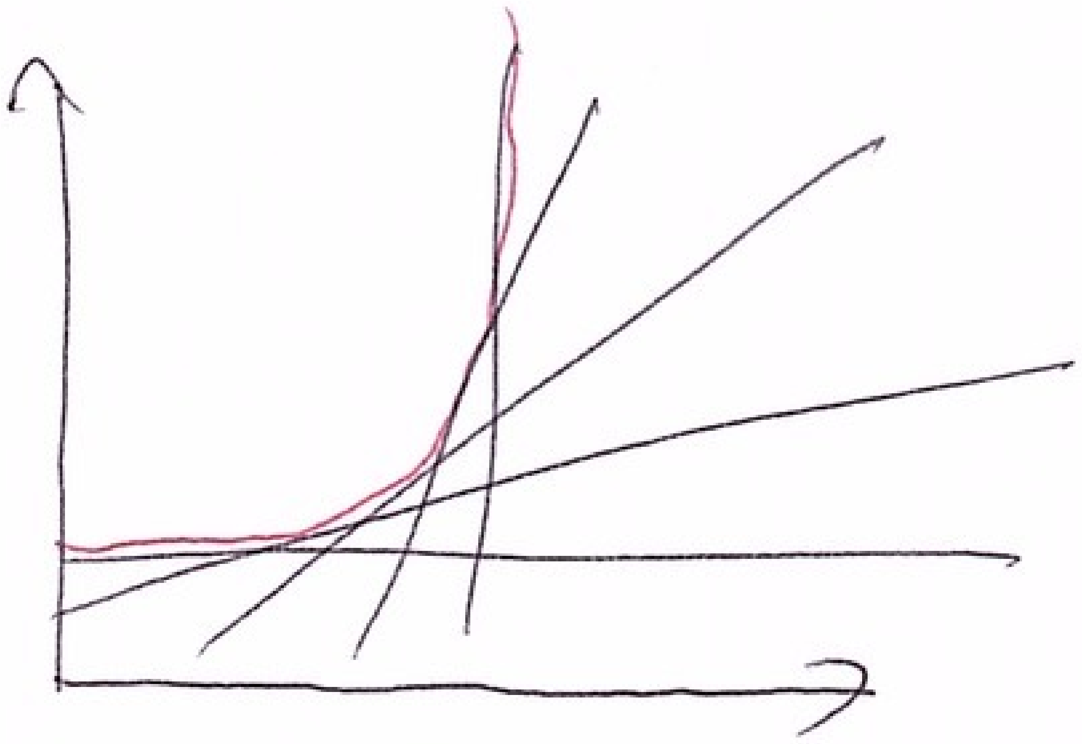
$$\overbrace{V(a)}^{\text{value function}} = \max_{b} \overbrace{v(a,b)}^{\text{objective function}} = v(a, \underbrace{b(a)}_{\text{policy function}})$$

$a$: state; $b$: choice

What is $V'(a)$?

An <u>upper envelope</u>: $V$ is ~~a~~ the upper envelope of $v(\cdot, b)$ (one for each $b$).



**Theorem (Envelope theorem)**

Let $v : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function ($\mathbb{R}^n$: state variables; $\mathbb{R}^m$: choice variables), and let $V(a) = \max_{b \in \mathbb{R}^m} v(a,b)$ be its value function, and let $b(a)$ be the policy function. If $V$ is differentiable, then

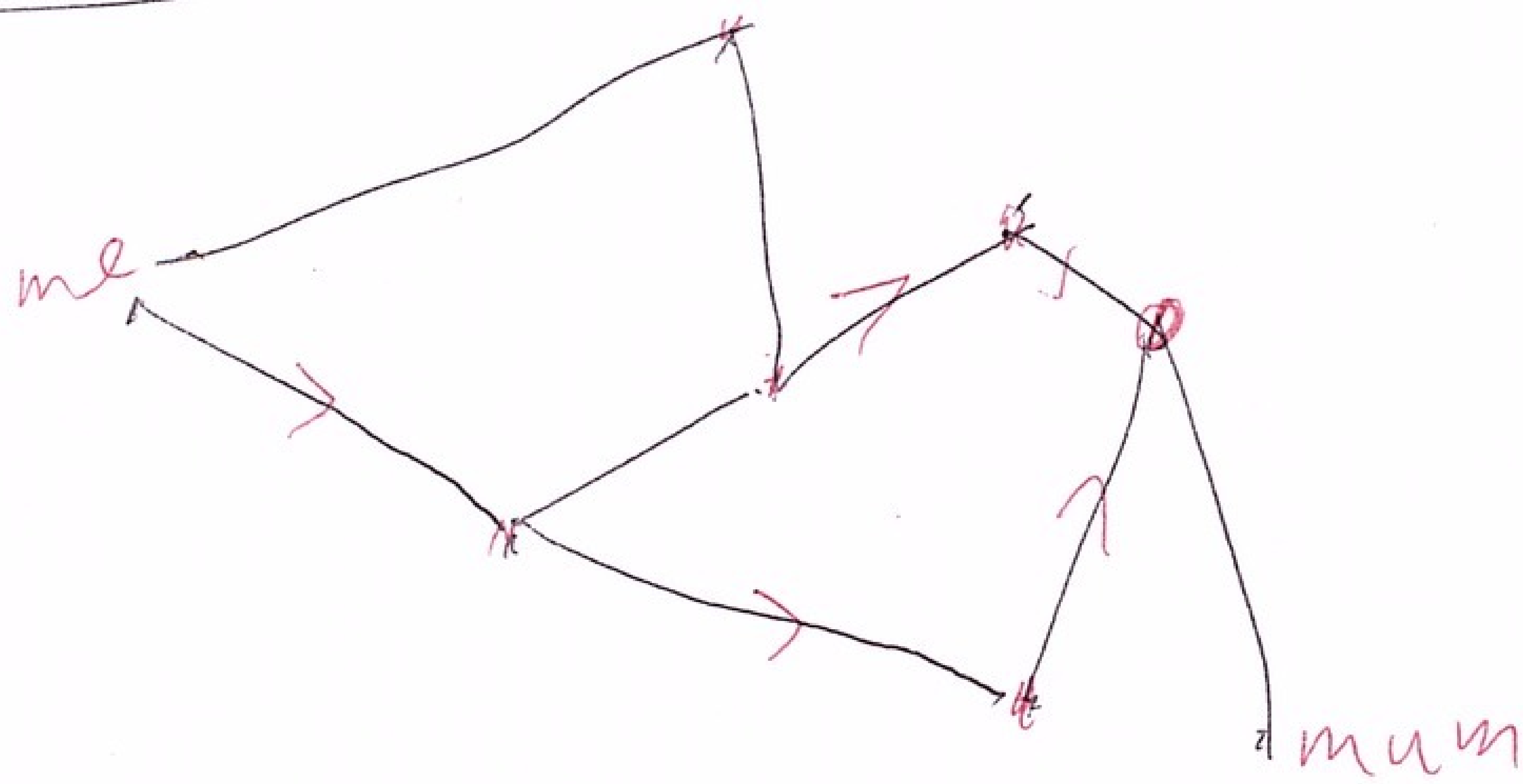
$$V'(a) = \left.\frac{\partial v(a,b)}{\partial a}\right|_{b=b(a)}$$

or in alternative notation,

$$V'(a) = v_a(a, b(a)).$$

# 2.4 Cost functions & Dynamic programming

## network




## gene sequences

A C G T ← "base pairs"

C C A G C C T A C red eyes

C C G G C C A A AC blue eyes

Needleman and Wunsch (1970)

## macro

$c_t$, $a_t$

↓

c today and a today.

$$\pi(p;w) = \max_{x \in \mathbb{R}^{N-1}_+} pf(x) - w \cdot x$$

value function

value function

$$\boxed{\pi(p;w) = \max_{y \in \mathbb{R}_+} py - c(y;w)}$$

where $\underline{c(y;w)} = \min_{x \in \mathbb{R}^{N-1}} w \cdot x$

s.t. $f(x) \geq y$.

cost function

Bellman equation

<u>Lemma</u> (Principal of optimality)

The Bellman equation holds true (for the firm's profit function).

<u>Proof</u>

$$\pi(p;w) = \max_{x \in \mathbb{R}^{N-1}_+} pf(x) - w \cdot x$$

$$= \max_{y \in \mathbb{R}_+, x \in \mathbb{R}^{N-1}_+} pf(x) - w \cdot x$$

"slack" choice

s.t. $f(x) = y$

$$= \max_{y \in \mathbb{R}_+} \left[ \begin{array}{ll} \max_{x \in \mathbb{R}^{N-1}_+} & pf(x) - w \cdot x \\ \text{s.t.} & f(x) = y \end{array} \right]$$

split max

substitution

$$= \max_{y \in \mathbb{R}_+} \left[ \begin{array}{ll} \max_{x \in \mathbb{R}^{N-1}_+} & py - w \cdot x \\ \text{s.t.} & f(x) = y \end{array} \right]$$

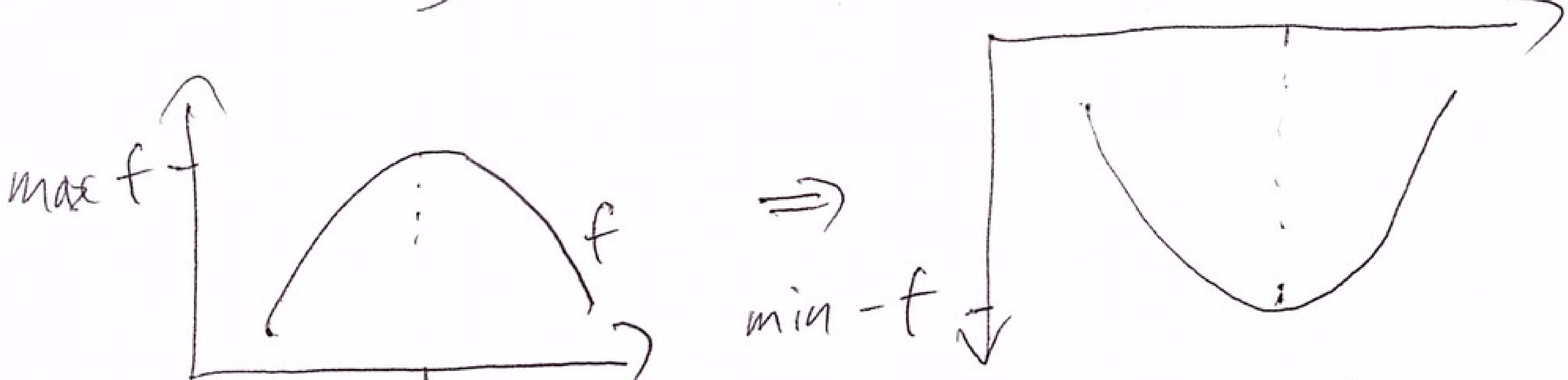
extracted a constant

$$= \max_{y \in \mathbb{R}_+} py + \left[ \begin{array}{ll} \max\limits_{x \in \mathbb{R}^{N-1}_+} & -w \cdot x \\ \text{s.t.} & f(x) = y \end{array} \right]$$

$$= \max_{y \in \mathbb{R}_+} py - \left[ \begin{array}{ll} \min\limits_{x \in \mathbb{R}^{N-1}} & w \cdot x \\ \text{s.t.} & f(x) = y \end{array} \right]$$

flip upside - down

$$= \max_{y} py - c(y;w). \quad \square$$




<u>claim</u>: $-\max\limits_{x} f(x) = \min\limits_{x} -f(x)$.

Bellman equation terminology

$$\pi(p;w) = \max_{y \in \mathbb{R}_+} py - c(y;w)$$

Bellman equation

state variable

value functions

choice variable

$$c(y;w) = \begin{array}{ll} \min\limits_{x \in \mathbb{R}^{N-1}_+} & w \cdot x \\ f(x) \geq y. & \end{array}$$

choice variable

state variables

**Theorem** For all prices $(p; w)$,

$$p = \left. \frac{\partial c(y;w)}{\partial y} \right|_{y = y(p;w)}.$$

**Proof** This is the FOC of the firm's Bellman equation. $\square$

Reapply envelope theorem to the Bellman equation:

$$\frac{\partial \pi(p;w)}{\partial p} = \left[ \frac{\partial}{\partial p} (py - c(y,w)) \right]_{y=y(p;w)}$$

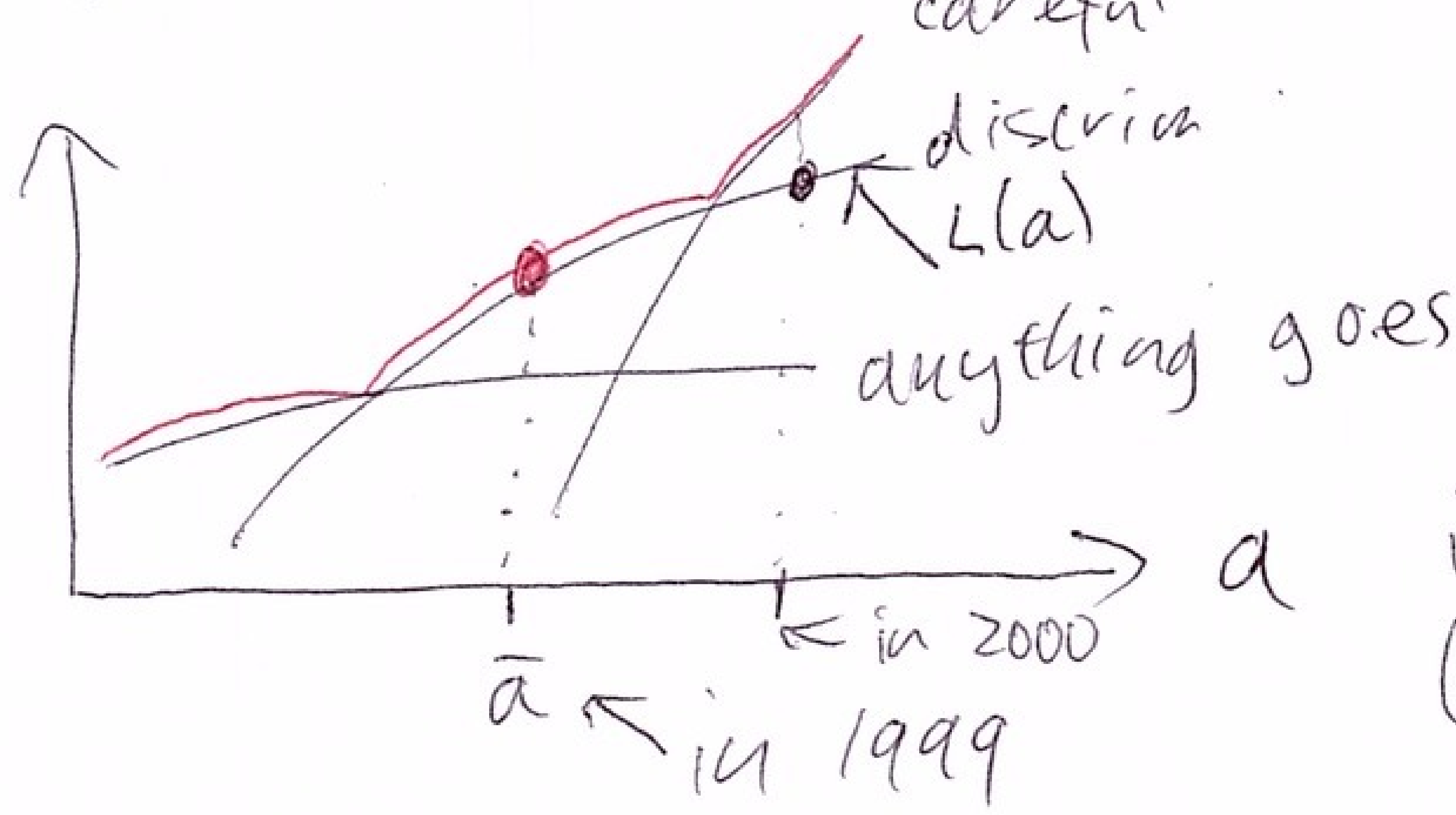
$$= y(p;w)$$

$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[ \frac{\partial}{\partial w_i} (py - c(y,w)) \right]_{y=y(p;w)}$$

$$= \left[ - \frac{\partial c(y,w)}{\partial w_i} \right]_{y=y(p;w)}.$$

<u>Proof</u> Fix a particular state $\bar{a}$.
The <u>lazy value function</u> for $\bar{a}$ is

$$L(a) = v(a, b(\bar{a})).$$




interest rates (demographics, etc.)

Observations:

* $L(a) \leq V(a)$ for all $a$.
* $L'(a) = v_a(a, b(\bar{a}))$.

Geometrically: red and black curves are tangent $\Rightarrow$ same derivatives.

$\bar{a}$ solves $\min_a V(a) - L(a)$

$$\text{FOC:} \quad V'(\bar{a}) - L'(\bar{a}) = 0$$
$$\Leftrightarrow V'(\bar{a}) = L'(\bar{a}).$$
$$= v_a(a, b(\bar{a})) \qquad \square$$

~~Chain-rule proof:~~

$$V(a) = v(a, b(a))$$
$$V'(a) = \left[v_a(a,b) + v_b(a,b)\, b'(a)\right]\Big|_{b=b(a)}$$
$$= \underbrace{v_a(a, b(a))}_{\text{"direct effect"}} + \underbrace{v_b(a, b(a))\, b'(a)}_{=0}$$

"indirect effect"

since $b(a)$ maximises $v(a,b)$,
$\rightarrow$ FOC $v_b(a, b(a)) = 0$. $\square$

Detour: $h(x) = f(g(x))$

and $g(x) = (x, x^2)$

$f(x, y) = x^2 + y.$

$h'(x) = f'(g(x))\, g'(x).$ (chain rule)

$$= \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2x \end{bmatrix}$$

$$= 4x.$$

$$\pi(w) = \max_{l} \; 10\sqrt{l} - wl$$

wages — price 10 — production function — labour costs

<u>with envelope theorem:</u>

$$\pi'(w) = \left[\frac{\partial(10\sqrt{l} - wl)}{\partial w}\right]_{l=l(w)}$$

$$= \left[-l\right]_{l=l(w)}$$

$$= -l(w).$$

<u>without envelope theorem:</u>

① Calculate $l(w)$:

FOC: $$10 \frac{1}{\sqrt{l}} \cdot \frac{1}{2} - w = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{l}} \cdot \frac{1}{2} = \frac{w}{10}$$

$$\Rightarrow \quad \sqrt{l} = \frac{5}{w}$$

$$\Rightarrow \quad l = \frac{25}{w^2}.$$

② Eliminate the max:

$$\pi(w) = 10\sqrt{l(w)} - w\,l(w)$$

$$= 10\sqrt{\frac{25}{w^2}} - w\left(\frac{25}{w^2}\right)$$

$$= 10 \cdot \frac{5}{w} - \frac{25}{w}$$

$$= \frac{25}{w} = 25w^{-1}$$

③ Differentiate $\pi$: $\pi'(w) = 25\,(-1)\,w^{-2}$

$$\pi'(w) = -\frac{25}{w^2} = -\ell(w).$$

Applying the envelope theorem to

$$\pi(p;w) = \max_{x \in \mathbb{R}^{N-1}_+} pf(x) - w \cdot x,$$

we get:

$$\frac{\partial \pi(p;w)}{\partial p} = \left[\frac{\partial}{\partial p}\left(pf(x) - w \cdot x\right)\right]_{x = x(p;w)}$$

$$= \left[f(x)\right]_{x = x(p;w)}$$

$$= f(x(p;w))$$

$$= y(p;w). \leftarrow \text{output "policy"}$$

$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\left(pf(x) - w \cdot x\right)\right]_{x = x(p;w)}$$

$$= \left[-x_i\right]_{x = x(p;w)}$$

$$= -x_i(p;w).$$

We can learn about the policy function by differentiating both sides:

$$\frac{\partial^2 \pi(p;w)}{\partial w_i^2} = -\frac{\partial}{\partial w_i} x_i(p;w).$$

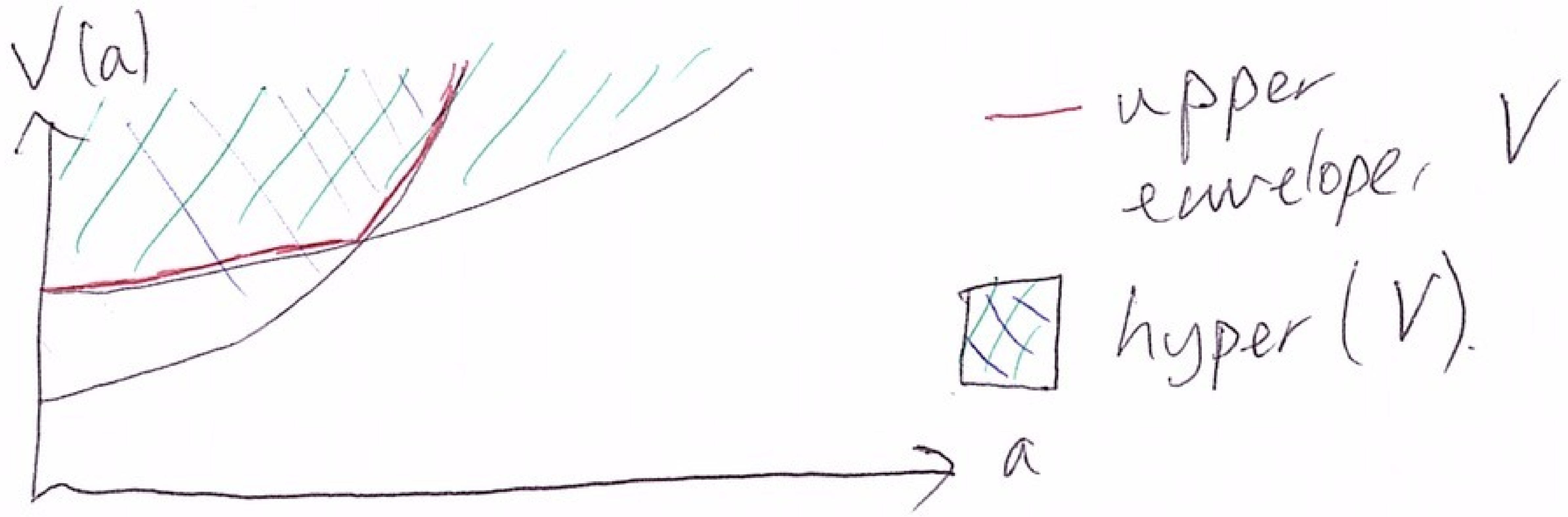
Also: $\frac{\partial^2 \pi(p;w)}{\partial p^2} = \frac{\partial y(p;w)}{\partial p}$

<u>Theorem</u> Suppose $V$ is the upper envelope of convex functions, specifically

$$V(a) = \max_b v(a,b)$$

where each $v(\cdot, b)$ is a convex function. Then $V$ is a convex function.

<u>Proof</u>



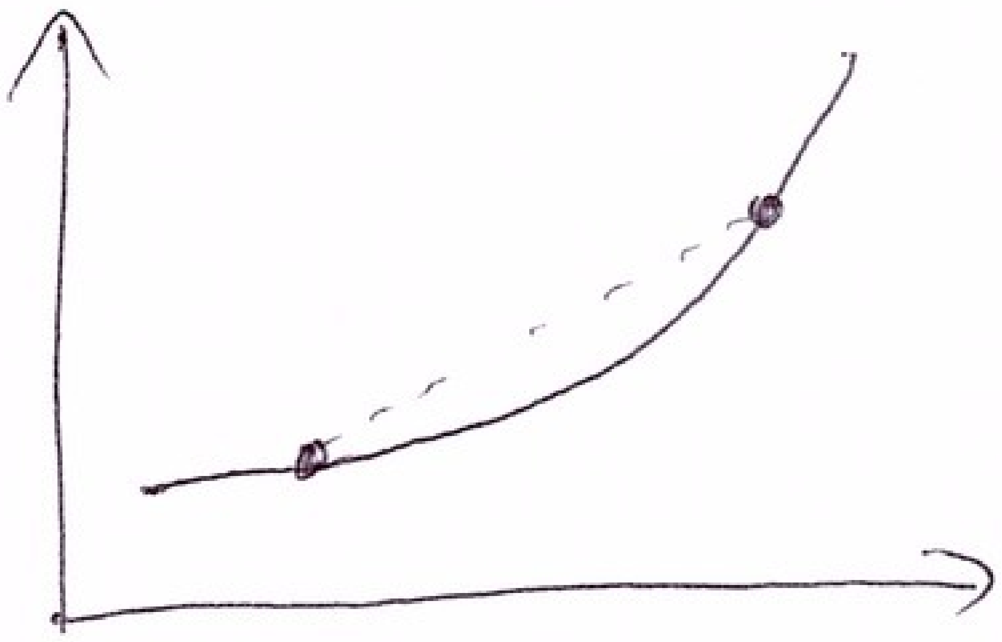

Geometric proof:

$* \; \text{hyper}(V) = \underbrace{\bigcap_b \underbrace{\text{hyper}(v(\cdot, b))}_{\text{all convex}}}_{\text{convex}}$

$*$ $\Rightarrow$ □

## Algebraic proof:

We would we like to prove that



convex: "line above curve".

$$\underbrace{t\,V(a) + (1-t)V(a')}_{\text{"line"}} \geq \underbrace{V(ta + (1-t)a')}_{\text{"curve"}}$$

for all $a, a'$, and $t \in (0,1)$.

$$
\begin{aligned}
t\,V(a) &+ (1-t)\,V(a') \\
&= t\,v(a, \underline{b(a)}) + (1-t)\,v(a', b(a')) \\
&\geq t\,v(a, \underbrace{b(a'')}_{\text{suboptimal choice}}) + (1-t)\,v(a', b(a')) \quad a'' = ta + (1-t)a' \\
&\geq t\,v(a, b(a'')) + (1-t)\,v(a', b(a'')) \\
&\geq v(a'', b(a'')) \quad \text{since } v(\cdot, b(a'')) \text{ is convex} \\
&= V(ta + (1-t)a').
\end{aligned}
$$

$\square$

**Theorem** For every production function $f$ [it need not be concave!] the firm's profit function is convex, and therefore

$$\frac{\partial y(p, w)}{\partial p} \geq 0 \quad \text{and} \quad \frac{\partial x_i(p, w)}{\partial w_i} \leq 0.$$

Proof:

$$\pi(p; w) = \max_{x \in \mathbb{R}^{N-1}_{+}} v(p, w; x)$$

where $v(p, w; x) = [p \;\; w] \begin{bmatrix} f(x) \\ -x \end{bmatrix}$,

and $v$ is linear in $(p, w)$.

Since linear functions are convex, we conclude that $v(\cdot, \cdot; x)$ is convex for all $x$.

By the theorem, $\pi$ is a convex function.

Therefore, $\frac{\partial^2 \pi}{\partial p^2}$ and $\frac{\partial^2 \pi}{\partial w_i^2}$ are $\geq 0$.

We conclude

$$\frac{\partial y(p; w)}{\partial p} = \frac{\partial^2 \pi(p; w)}{\partial p^2} \geq 0$$

and $$\frac{\partial x_i(p; w)}{\partial w_i} = -\frac{\partial^2 \pi(p; w)}{\partial w_i^2} \leq 0. \quad \square$$

Marginal values vs FOC's:

$$\pi(p; w) = \max_{l} \; p\sqrt{l} - wl$$

value function

marginal value: $\frac{\partial \pi}{\partial p} = \sqrt{l(p;w)} = y(p;w)$ $\boxed{\geq 0}$

state

(envelope theorem)

(since $\pi$ is convex)

FOC: $\frac{1}{2}\frac{p}{\sqrt{l}} = w \iff \frac{2\sqrt{l}}{p} = \frac{1}{w}$

objective function

$$\frac{\partial}{\partial l}\left[p\sqrt{l} - wl\right] \boxed{= 0}$$

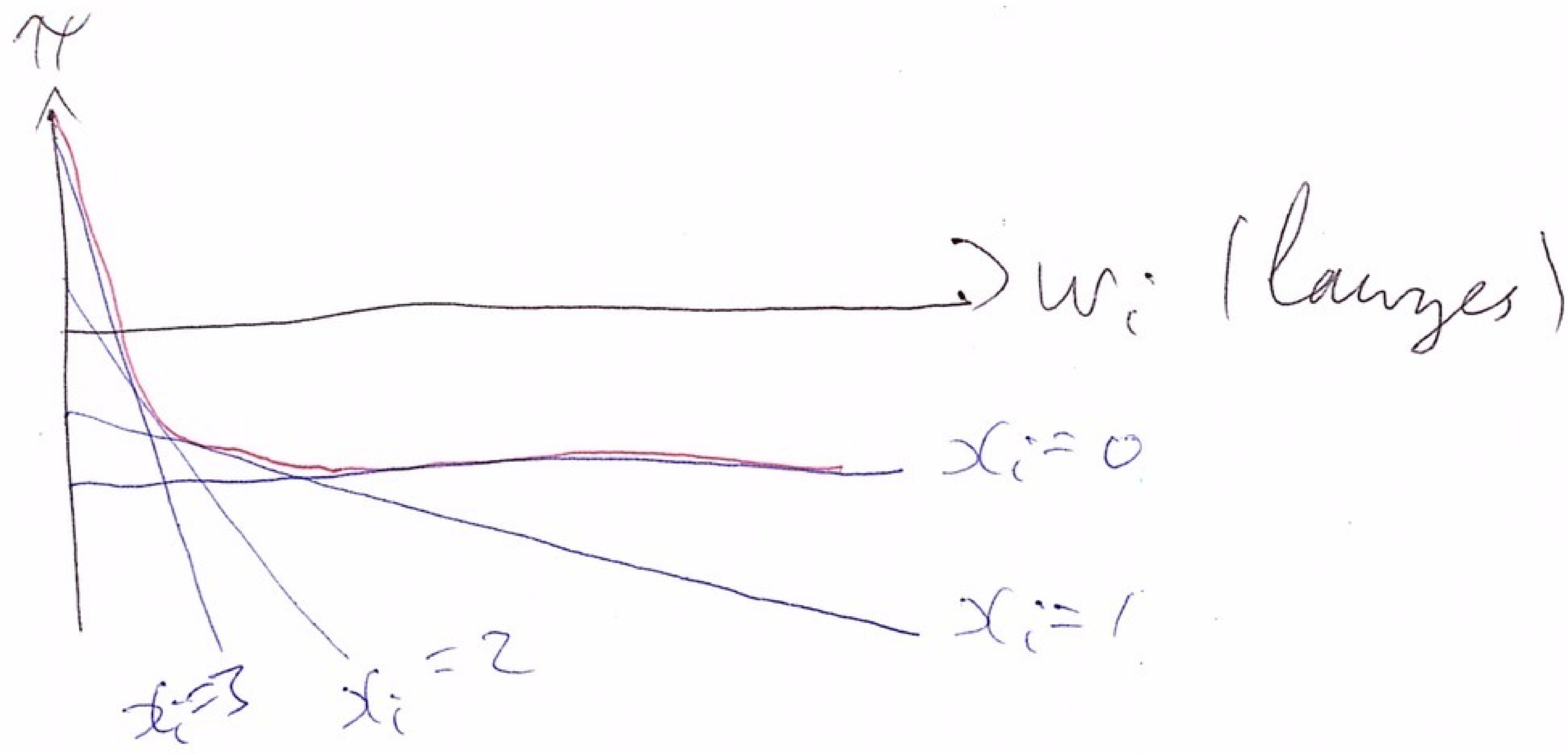
↑ choice variable

$\iff \sqrt{l} = \frac{p}{2w}$

$\iff l = \frac{p^2}{4w^2}.$

(first-order conditions)

* state vs choice
* value vs objective
* $\geq 0$ vs $= 0$.




— is convex

because prices are linear
(<u>not</u> because f is concave)