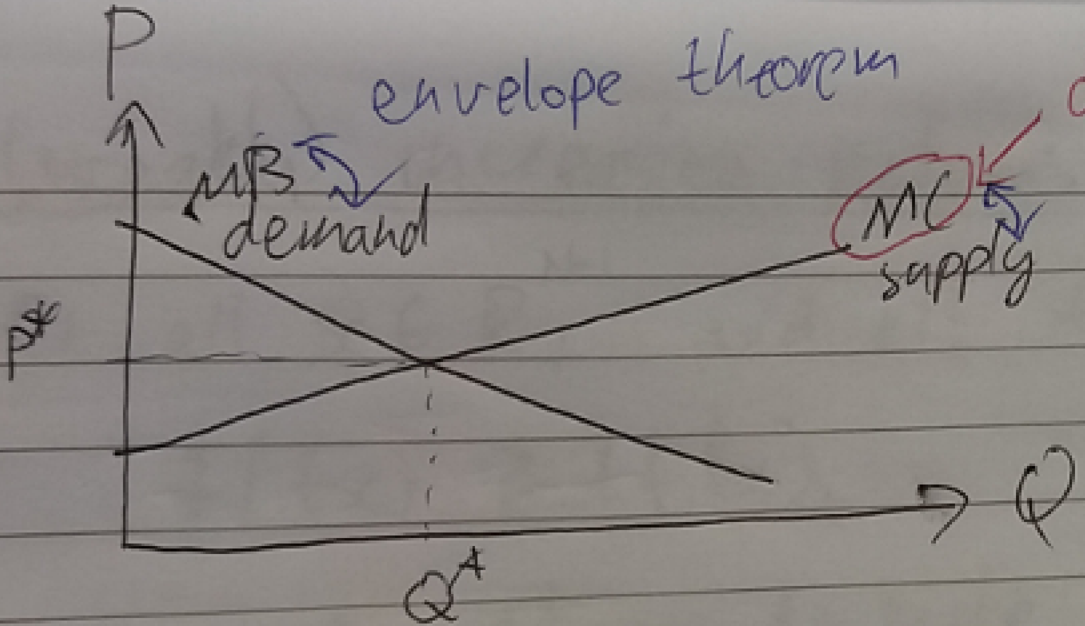
envelope theorem

dynamic programming

envelope theorem



MC ↑ convex analysis

Chapter 2 - firms

competitive firms - price takers

N # goods in the economy

$N-1$ inputs

1 output

$y = f(x)$, $f: \mathbb{R}_+^{N-1} \to \mathbb{R}_+$

output input

* inaction: $f(0) = 0$ "boring"

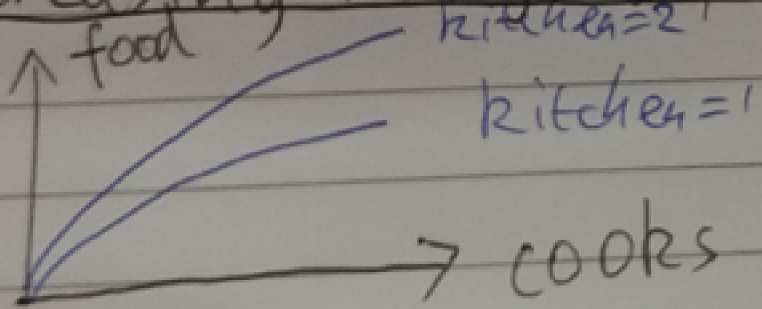
* inaction: $f(0) = 0$

"boring"

* free disposal / monotonicity:
  If $x \geq x'$, i.e. $x_i \geq x_i'$ for each good
  then $f(x) \geq f(x')$.

* smoothness: f is twice differentiable

* decreasing marginal productivity:



* (weakly) increasing returns to scale:

for all $x \in \mathbb{R}^{N-1}_{+}$ and all $t > 1$,

$$f(tx) \geq t f(x).$$

* constant returns to scale:

for all $x \in \mathbb{R}^{N-1}_{+}$ and all $t > 1$,

$$f(tx) = t f(x).$$

* (weakly) decreasing RTS:

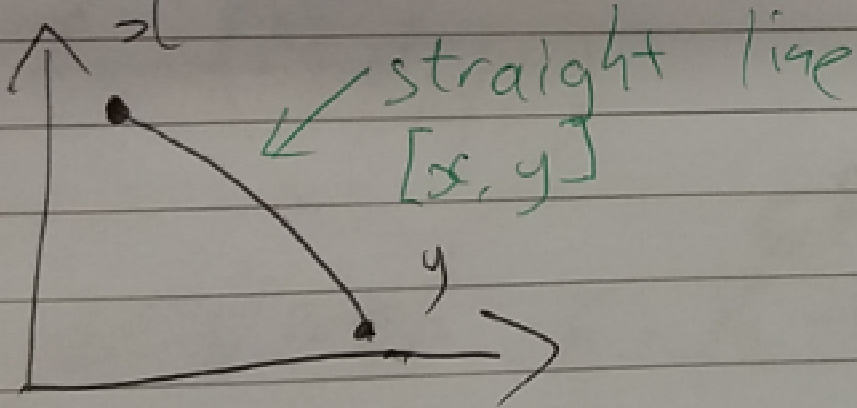
for all $x \in \mathbb{R}^{N-1}_{+}$ and all $t > 1$

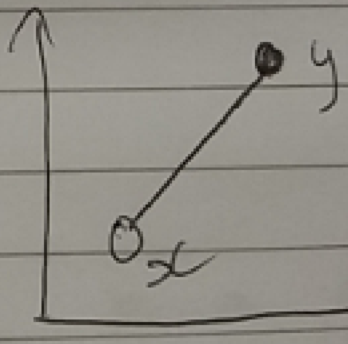
$$f(tx) \le t f(x).$$

short cut assumption for capturing misspecification of leaving out a factor of production or treating different goods as the same

Appendix D - Convex geometry

Def D.1 A closed interval between two points $x, y \in \mathbb{R}^n$ is $[x,y] = \{\overbrace{ax+(1-a)y}^{\text{convex combination of } x \text{ and } y} : a \in [0,1]\}$



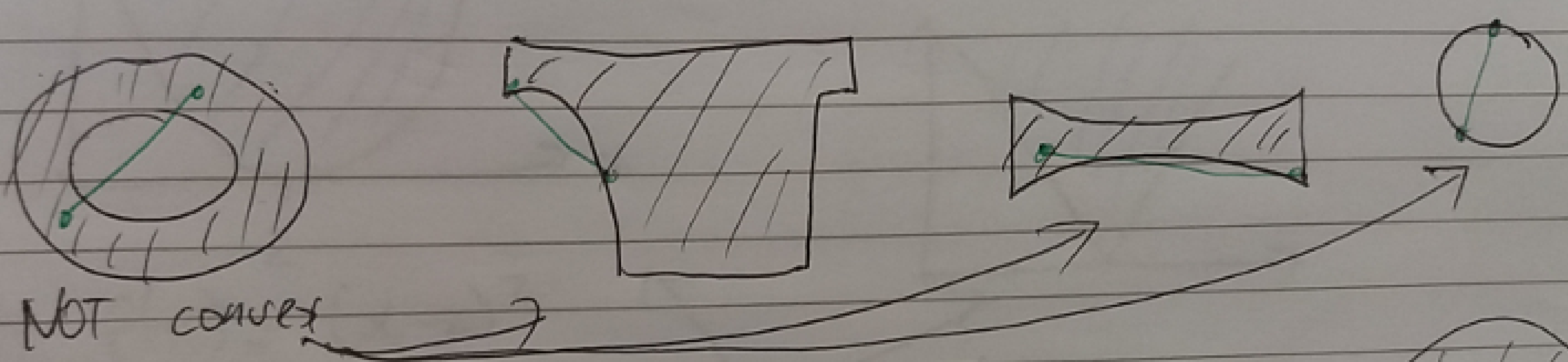


$(x, y]$

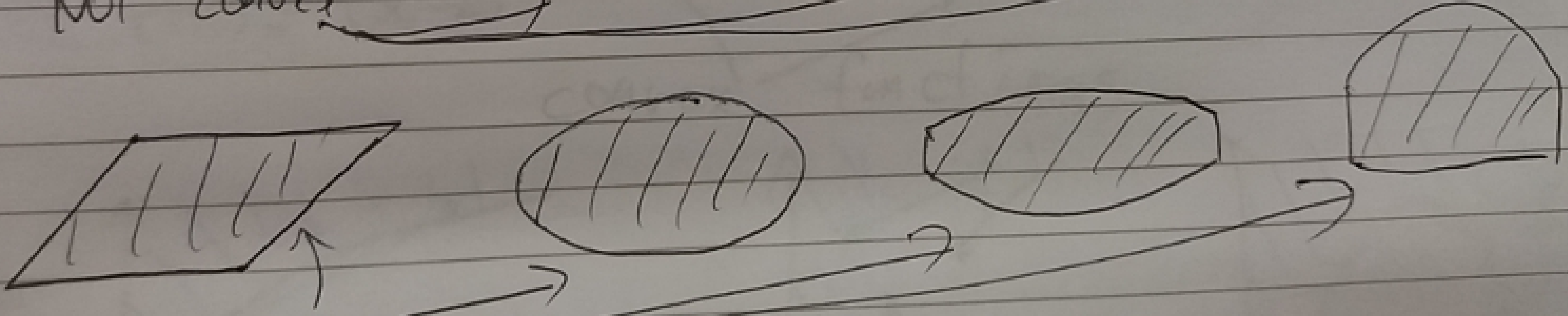
means $y \in (x, y]$

Def D.2 $X \subseteq \mathbb{R}^n$ is a convex set if for all $x, y \in X$, then $[x,y] \subseteq X$.

$x \notin [x,y]$

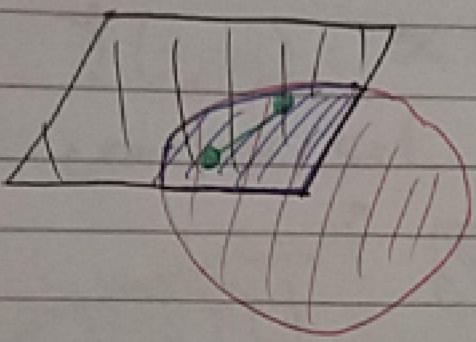


NOT convex



convex

<u>Theorem</u> The intersection of convex sets is convex. eg: If A and B are convex sets then $A \cap B$ is a convex set

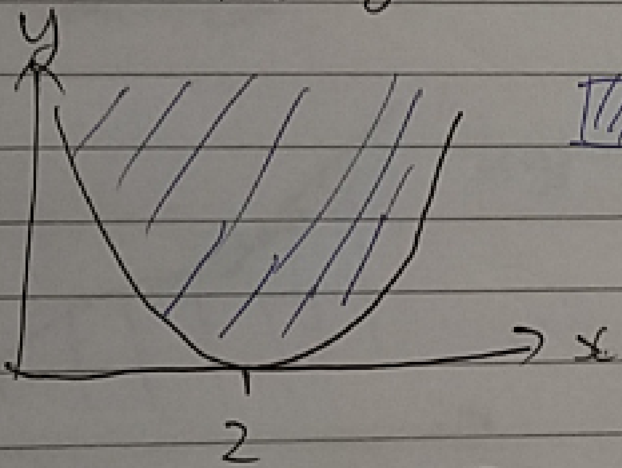


$\subseteq \mathbb{R}^n$

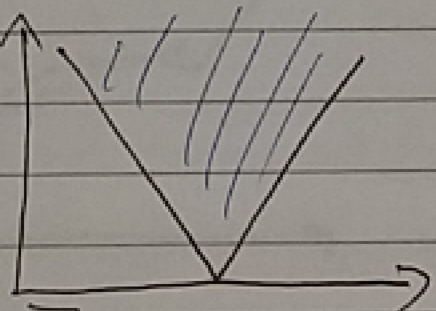
<u>Def</u> $f: X \to \mathbb{R}$ is a <u>convex functi</u>
if its <u>hypergraph</u>

$\{(x,y) : x \in X,\ y \geq f(x)\} = \text{hyper}(f)$

is a convex s

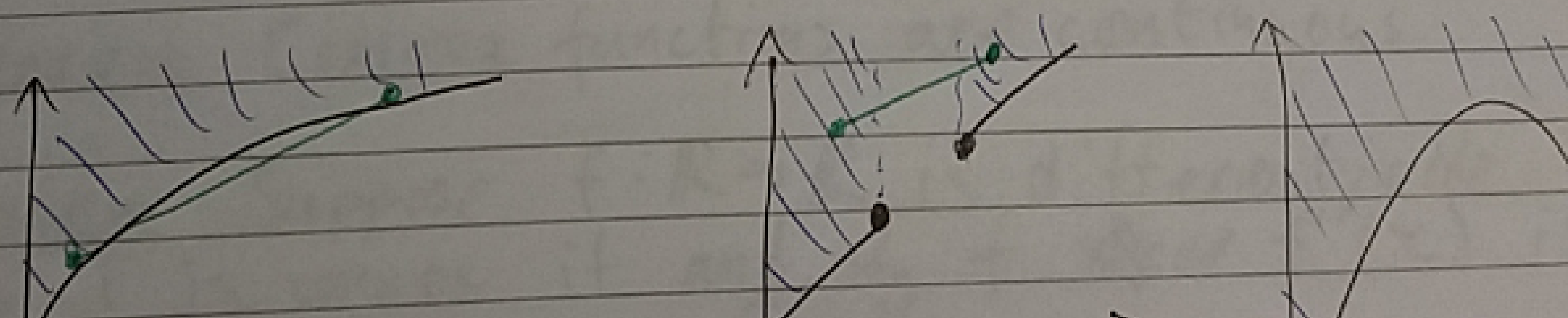


hyper(f)



$f(x) = (x-2)^2$

convex functions



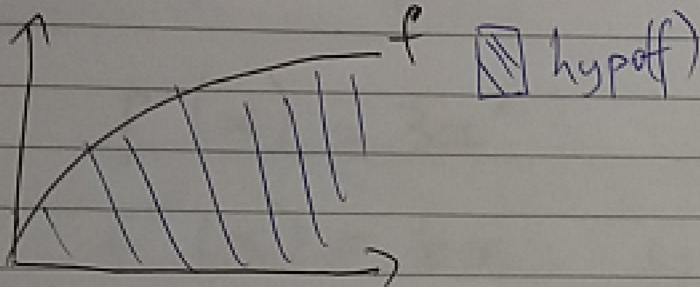
NOT convex

Def $f: X \to \mathbb{R}$ is a <u>concave function</u> if $-f$ is a convex function.

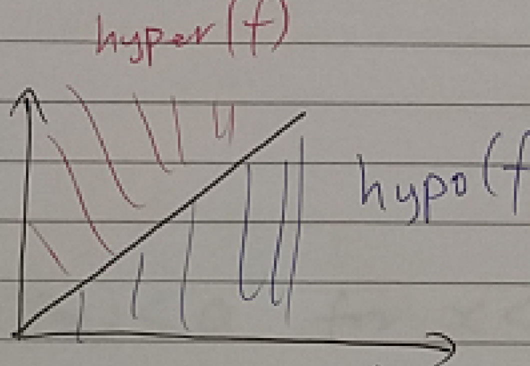
Equivalently, f is concave if its hypograph

$$\{(x,y) : x \in X,\ y \leq f(x)\} = \text{hypo}(f)$$

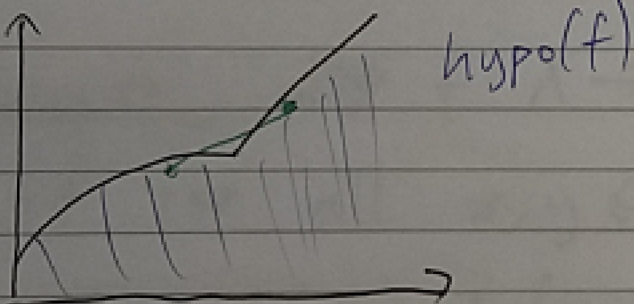
is a convex set.



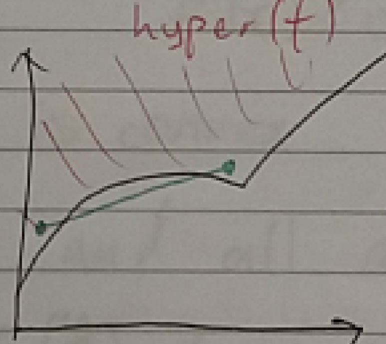
Concave function



concave and convex



neither concave nor convex



<u>Theorem</u> Convex functions are continuous.

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is convex if and only if $f'(x)$ is a weakly increasing function.

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is convex iff (if and only if) $f''(x) \geq$ for all x.

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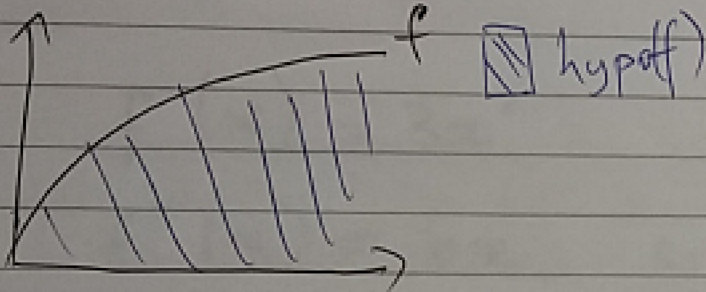
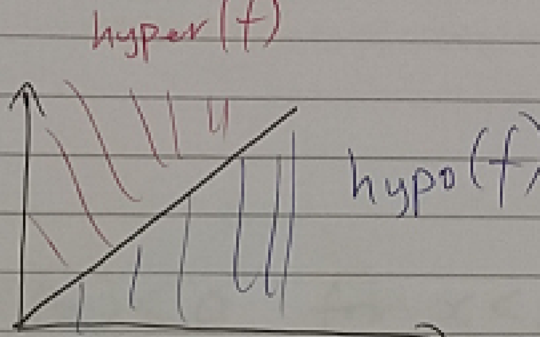
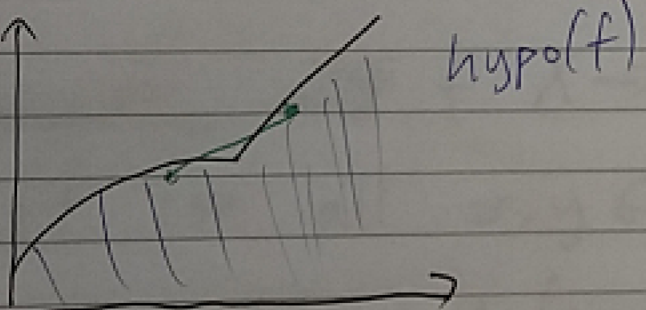


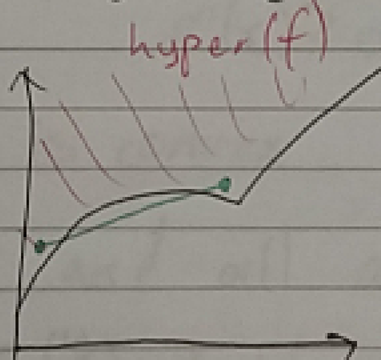
concave function



concave and convex



neither concave nor convex



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back to Chapter 2

* Concavity: f is a concave function.

Claim If f is ~~smooth~~ concave, and allows inaction then f has (weakly) decreasing returns to scale.

Proof: We want to show $f(tx) \leq t f(x)$, for $t \geq 1$.
Let $a = \frac{1}{t}$. Notice that $a \in [0,1]$.
By a theorem (D.6), [points: tx and 0]

$$a\, f(tx) + (1-a) f(0) \leq f(a t x + (1-a) 0)$$

$$\Rightarrow \frac{1}{t} f(tx) + \cancel{(1-\tfrac{1}{t}) f(0)} \leq f(\tfrac{1}{t} t x + \cancel{(1-\tfrac{1}{t}) 0})$$

$$\Rightarrow \frac{1}{t} f(tx) \leq f(x)$$

$$\Rightarrow f(tx) \leq t f(x). \quad \square$$

Claim If f is smooth and concave, then it has (weakly) decreasing marginal productivity.

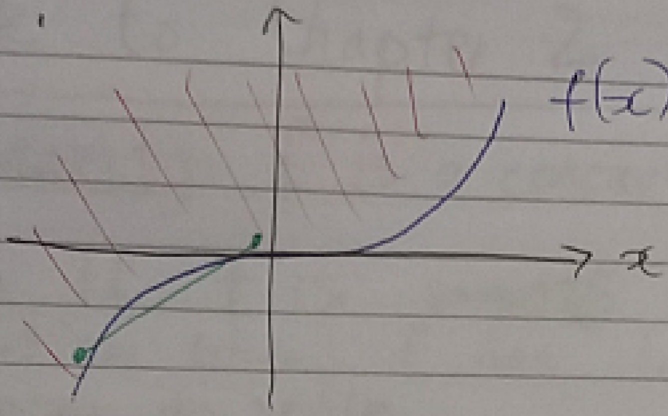
back to Appendix D.

Def The upper contour set of a function $f: X \to \mathbb{R}$ at level u is

$$\{x \in X : f(x) \geq u\}.$$

deceptively similar to "hyper graph"

eg



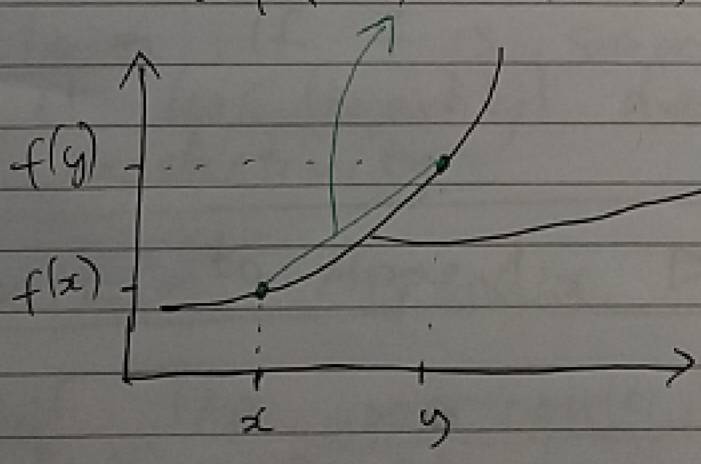
$f'(x) = 3x^2$

$f''(x) = 6x$, so $f''(x) < 0$ for $x < 0$.

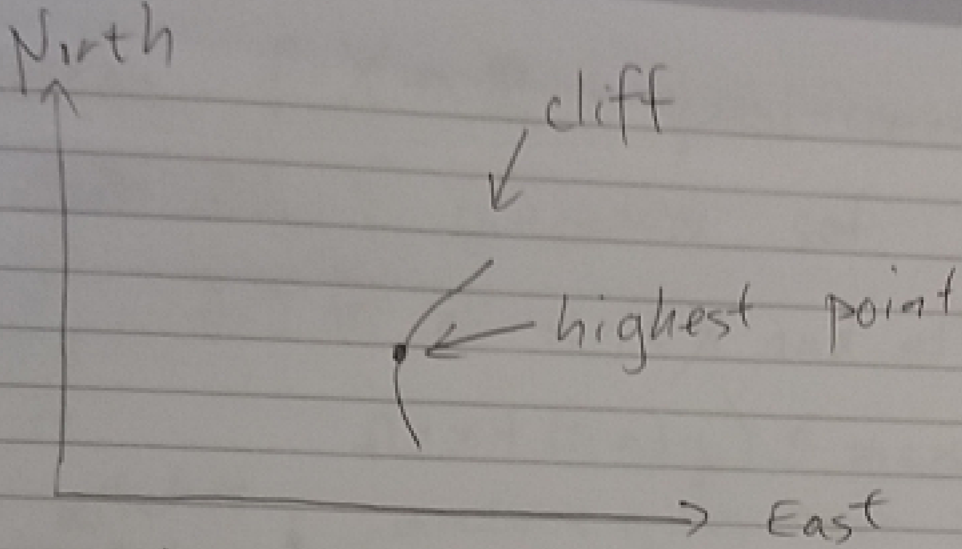
NOT convex but also, NOT true that $f''(x) \geq 0$ for all x.

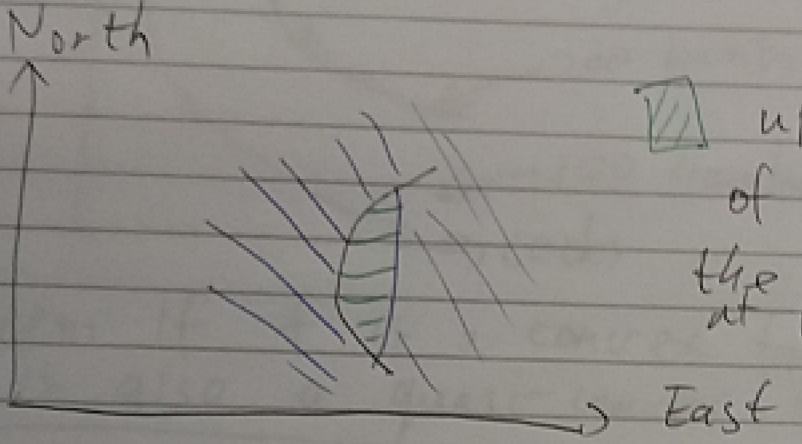
Theorem ~~Acf~~ $f : X \to \mathbb{R}$ is convex iff for all $x, y \in X$ and all $a \in (0,1)$,

$$a f(x) + (1-a) f(y) \geq f(ax + (1-a)y).$$



"line is above the curve"

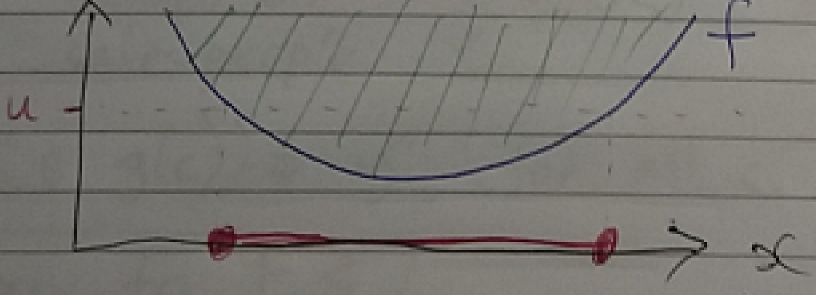




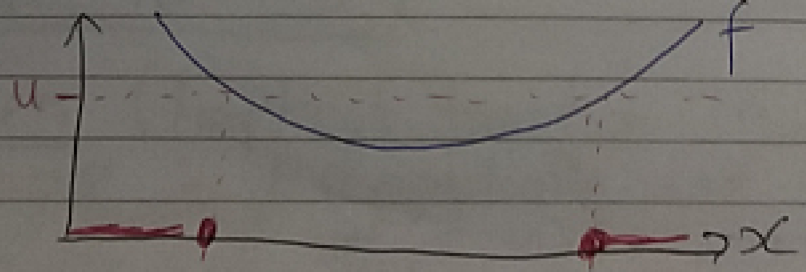
upper contour set of $h(x)$ which give the height (altitude) at location x.

lower contour set

<u>Def</u> $f: X \to \mathbb{R}$ is a <u>quasi-convex function</u> if all of its lower contour sets are convex
f is a <u>quasi-concave function</u> if all of its upper contour sets are convex sets.



lower ~~upper~~ contour set of f at level u



hyper(f)

upper contour set at level u

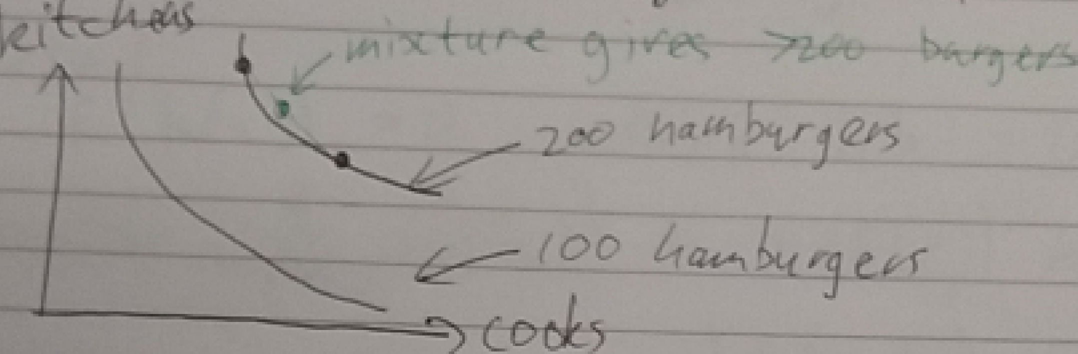
Theorem $f: X \to \mathbb{R}$ is quasi-convex

$\Longleftrightarrow$ (i) X is a convex set

if and only if

(ii) for all $x, y \in X$ and all $a \in (0,1)$,

$$f(ax + (1-a)y) \leq \max\{f(x), f(y)\}$$



Theorem If f is a convex function then f is also a quasi-convex function.

Back to Chapter 2

* quasiconcavity: f is a quasi-concave function, i.e the upper contour set for each output target y is a convex set.

c computers $y = f(c,s)$ sales
s salespeople

$$f(c, g(c)) = y^* \text{ for all } c.$$

Detour to F.3

Theorem (Implicit function theorem). Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $f(x, g(x)) = z^*$ for all x. If $\frac{\partial f}{\partial y}(x,y)\Big|_{y=g(x)} \neq 0$ then

$$g'(x) = -\frac{f_x(x,y)}{f_y(x,y)}\Big|_{y=g(x)}$$

Proof: Differentiate both sides of

$$f(x, g(x)) = 0.$$

This gives:

$$f_1(x, g(x)) + f_2(x, g(x))g'(x) = 0$$

Detail:

$f(h(x)) = 0$, where $h(x) = (x, g(x))$

Chain rule: $\frac{d}{dx} f(h(x)) = f'(h(x))\, h'(x)$

$$= [f_1(x, g(x)) \quad f_2(x, g(x))] \begin{bmatrix} 1 \\ g'(x) \end{bmatrix}$$

$$= f_1(x, g(x)) + f_2(x, g(x))\, g'(x)$$

Rearranging gives $g'(x) = -f_1(x, g(x)) / f_2(x, g(x))$. $\square$

Back to chapter 2

$$\underset{g'(c)}{MRS} = - \frac{\frac{\partial f(c,s)}{\partial c}}{\frac{\partial f(c,s)}{\partial s}} \Bigg|_{s=g(c)} = - \frac{MPC}{\cancel{M}MPS}.$$

2.2 Profit maximisation

$$\underset{\text{profit}}{\pi}(p; w) = \max_{x \in \mathbb{R}^{N-1}_{+}} \underbrace{pf(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{cost}}$$

p: output price

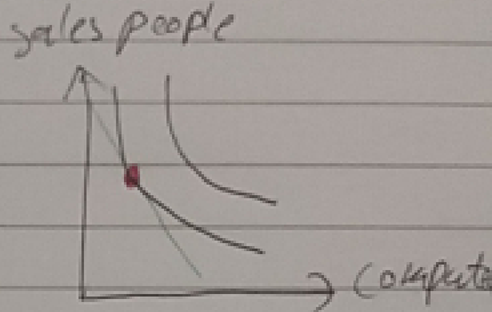
w: inputs prices, $w \in \mathbb{R}^{N-1}_{+}$

$$= pf(x(p;w)) - w \cdot x(p;w)$$

$\underbrace{x(p;w)}_{\text{factor demand function}}$

$\underline{\text{FOC } x_i}$: $\quad p \frac{\partial f}{\partial x_i}(x) = w_i$

$\frac{\text{FOC } x_i}{\text{FOC } x_j}$: $\quad \frac{\frac{\partial f(x)}{\partial x_i}}{\frac{\partial f(x)}{\partial x_j}} = \frac{w_i}{w_j}$.



i = computers

j = sales people