C.8 Fixed Points

<u>Def</u> A function f is a <u>self-map</u> if $f: X \to X$.

<u>Def</u> Let $f: X \to X$ be a self-map. We say $x^* \in X$ is a <u>fixed point</u> of f if $x^* = f(x^*)$.

In macro: $V(a) = \max_{c, a'} u(c) + \beta V(a')$.

$$V' = \Gamma(V)(a) = \max_{c, a'} u(c) + \beta V(a')$$

eg: $\Gamma(0)(a) = u(a)$

"real" V: $V^*(a_0) = \max_{c_t, a_{t+1}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$

s.t. $c_t + a_{t+1} = a_t \quad \forall t$.

<u>claim</u>: $V^* = \Gamma(V^*)$.

<u>Def</u> Consider (X, d_X), (Y, d_Y) and any $a > 0$. We say $f: X \to Y$ is <u>Lipschitz continuous of degree a</u> if for every $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq a\, d_X(x, x').$$

<u>Def</u> We say $f: X \to X$ is a <u>contraction of degree a</u> if ~~a~~ f is Lipschitz continuous ~~and~~ of degree $a < 1$.

<u>Theorem</u> (Banach's fixed point theorem)
Let (X, d) be a complete metric space. If $f: X \to X$ is a contraction of degree a, then:
(i) f has a unique fixed point x^*.
(ii) Given any $x_0 \in X$, then $x_{n+1} = f(x_n)$ converges to x^*.
(iii) $d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1)$.

<u>Proof</u> (i) <u>uniqueness:</u> Suppose $x^* \neq x^{**}$ and both are fixed points.
So $x^* = f(x^*)$ and $x^{**} = f(x^{**})$.
$\Rightarrow d(x^*, x^{**}) = d(f(x^*), f(x^{**}))$.
But since f is a contraction,
$$d(f(x^*), f(x^{**})) \leq a\, d(x^*, x^{**}) < d(x^*, x^{**}) \quad ↯$$

(i) <u>existence</u>: We will prove x_n is a Cauchy sequence. Repeatedly applying the contraction property, we get

$$d(x_n, x_{n+m}) \le a^n d(x_0, x_m)$$

$$\cancel{d(x_n, x_{n+1}) \le a\, d(}$$

$$d(x_1, x_{m+1}) \le a\, d(x_0, x_m)$$

$$\overbrace{d(f(x_0), f(x_m))}$$

$$d(x_2, x_{m+2}) \le a \underbrace{d(x_1, x_{m+1})}_{\le a\, d(x_0, x_m)} \le a^2 d(x_0, x_m)$$

This implies:

triangle inequality ("short-cut")

$$d(x_0, x_m) \le d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{m-1}, x_m)$$

$$\le d(x_0, x_1) + a\, d(x_0, x_1) + a^2 d(x_0, x_1) + \dots$$

using formula above

$$= \frac{1}{1-a} d(x_0, x_1) \leftarrow \text{geometric series}$$

Combining:

$$d(x_n, x_{n+m}) \le \frac{a^n}{1-a} d(x_0, x_1), \text{ for all } n, m$$

$$\Rightarrow (*)\ d(x_n, x_m) \le \frac{a^N}{1-a} d(x_0, x_1) \text{ for all } n, m \ge N.$$

So x_n is a Cauchy sequence.

Since (X, d) is complete, we conclude $x_n \to x^*$, for some $x^* \in X$.

Consider $y_n = f(x_n) = x_{n+1}$.

Since f is continuous, $y_n \to f(x^*)$.

Since $y_n = x_{n+1}$, $y_n \to x^*$. So $x^* = f(x^*)$

(since sequences can converge to at most one point.) Therefore, x^* is a fixed point of f.

(iii) Approximation bound: By continuous of d and Cauchy formula (*),

$$d(x_n, x^*) = \lim_{m\to\infty} d(x_n, x_m) \leq \frac{a^n}{1-a} d(x_0, x_1). \square$$

C.9 Compact Sets

Def Let A be a subset in (X, d). We say A is compact if for every sequence $x_n \in A$, there is a convergent subsequence $y_n \to y^*$ such that $y^* \in A$.

Theorem (Extreme Value Theorem)
Suppose $f: X \to \mathbb{R}$ is continuous. If (X, d) is a compact metric space, then f has a min and a max, i.e.

$$\max_{x \in X} f(x) \text{ has a solution.}$$

<u>Theorem</u> Suppose $f : X \to Y$ is continuous, and (X, d_X) is compact and $Y = f(X)$. Then (Y, d_Y) is compact.