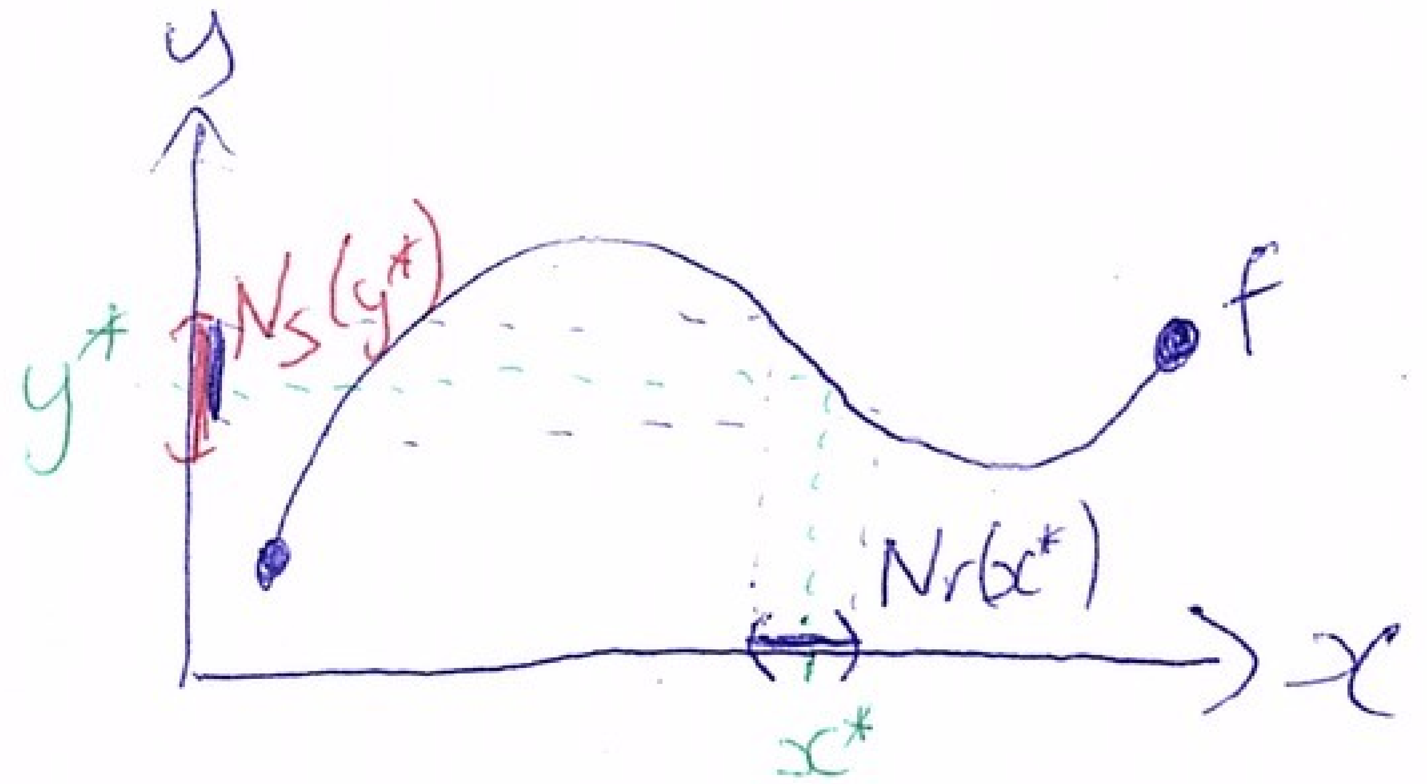
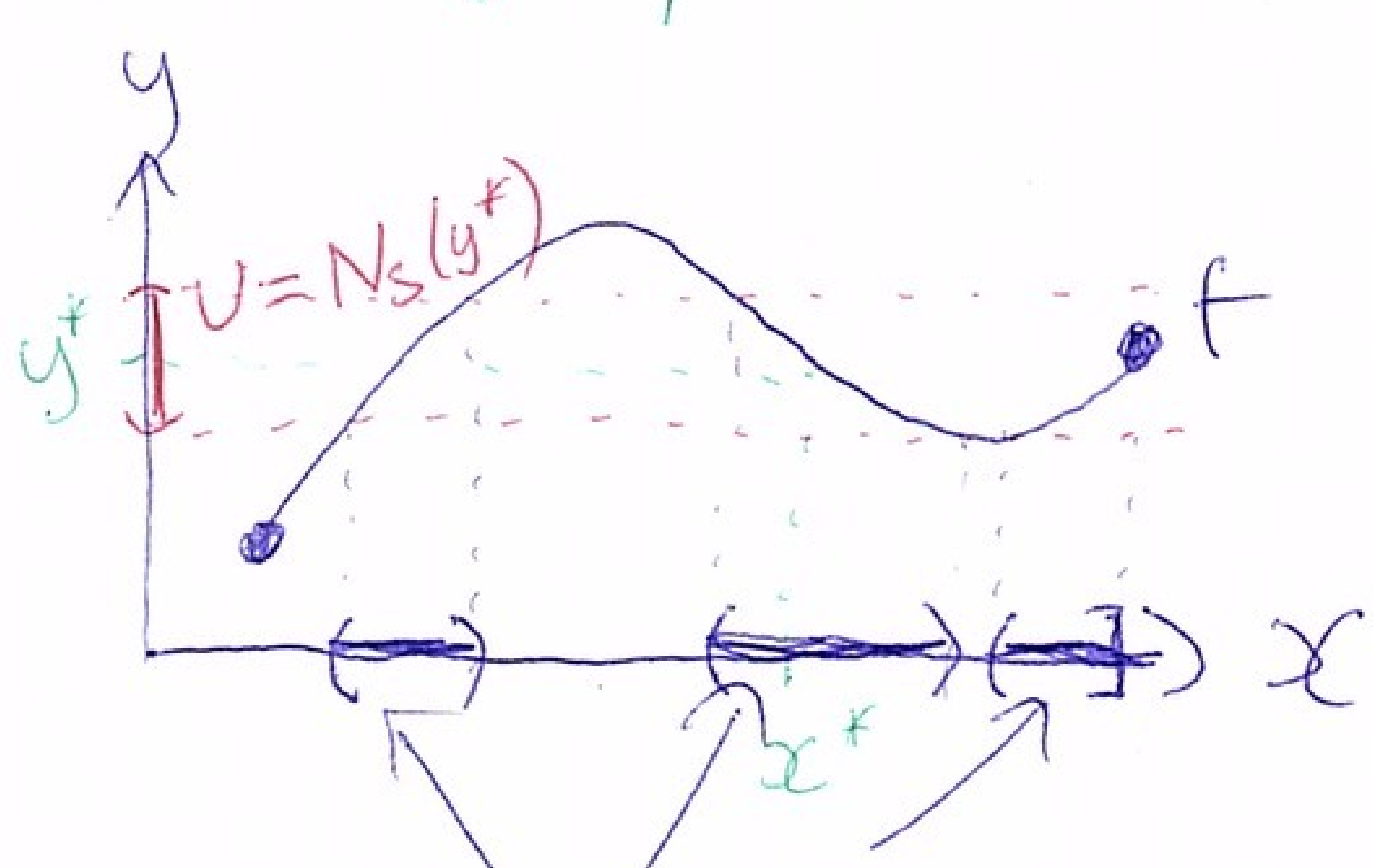
Theorem Let $f: X \to Y$ be a function between (X, d_X) and (Y, d_Y). Pick any $x^* \in X$, and let $y^* = f(x^*)$. Then f is continuous at x^* $\iff$ for every open ball $N_s(y^*)$, there exists an open ball $N_r(x^*)$ such that $f(N_r(x^*)) \subseteq N_s(y^*)$.

$\{f(x) : x \in N_r(x^*)\}$



Theorem Let $f: X \to Y$ be a function between (X, d_X) and (Y, d_Y). Then f is continuous $\iff$ for all open sets $U \subseteq Y$, $\underbrace{f^{-1}(U)}$ is an open set in (X, d_X).

"inverse image or pull back of U" $\{x \in X : f(x) \in U\}$



$f^{-1}(U)$ is an open set in (X, d_X)

C7 Completeness

<u>Def</u> Let (X, d) be a metric space.
We say $x_n \in X$ is a <u>Cauchy sequence</u> if for every $r > 0$, there exists an N such that for all $n, m > N$,

$$d(x_n, x_m) < r.$$

<u>Def</u> A metric space (X, d) is <u>complete</u> if every Cauchy sequence $x_n \in X$ is convergent.

e.g.
* $(\mathbb{R}, d_2)$ is complete.
* $((0,1], d_2)$ is <u>not</u> complete because $x_n = \frac{1}{n}$ is a Cauchy sequence, but x_n is not convergent.
* $(\mathbb{Q}, d_2)$ is <u>not</u> complete

<u>Theorem</u> Let (X, d) be any metric space.
If $x_n \in X$ is a convergent sequence, then x_n is a Cauchy sequence.

<u>Theorem</u> Let (X, d) be any metric space.
If $x_n \in X$ is a Cauchy sequence, and $y_n \to y^*$ is a subsequence of x_n, then $x_n \to y^*$.

Theorem $(\mathbb{R}, d_2)$ is a complete metric space.

Proof Two ideas from real analysis:

(i) every real sequence has a weakly monotone subsequence.

(ii) if a real sequence is bounded and weakly monotone, then it converges.

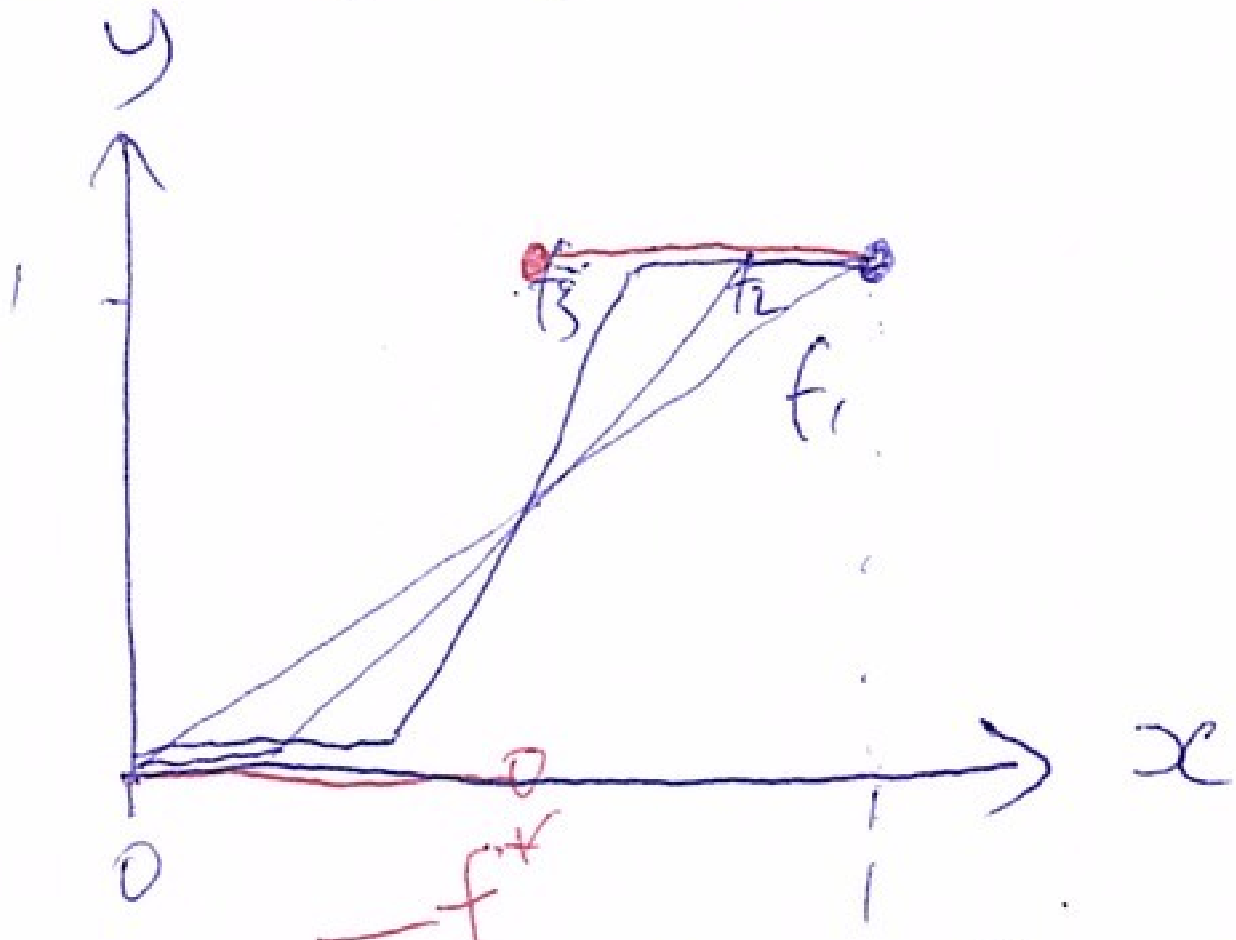
Let $x_n \in \mathbb{R}$ be any Cauchy sequence.
Let y_n be a monotone subsequence, by (i).
So y_n is a Cauchy sequence.
And y_n is a bounded sequence.
By (ii), ~~y_n~~ $y_n \to y^*$ for some $y^* \in \mathbb{R}$.
Therefore $x_n \to y^*$. $\square$

<u>Theorem</u> Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is a complete metric space, then $(\underbrace{B(X,Y)}_{\{f : X \to Y \text{ s.t. } f \text{ is bounded}\}}, d_\infty)$ and $(CB(X,Y), d_\infty)$ are complete metric spaces.

$CB(X,Y) = \{f \in B(X,Y) \text{ s.t. } f \text{ is continuous}\}$

bounded means $f(X) \subseteq N_r(y_0)$ for some (r, y_0)

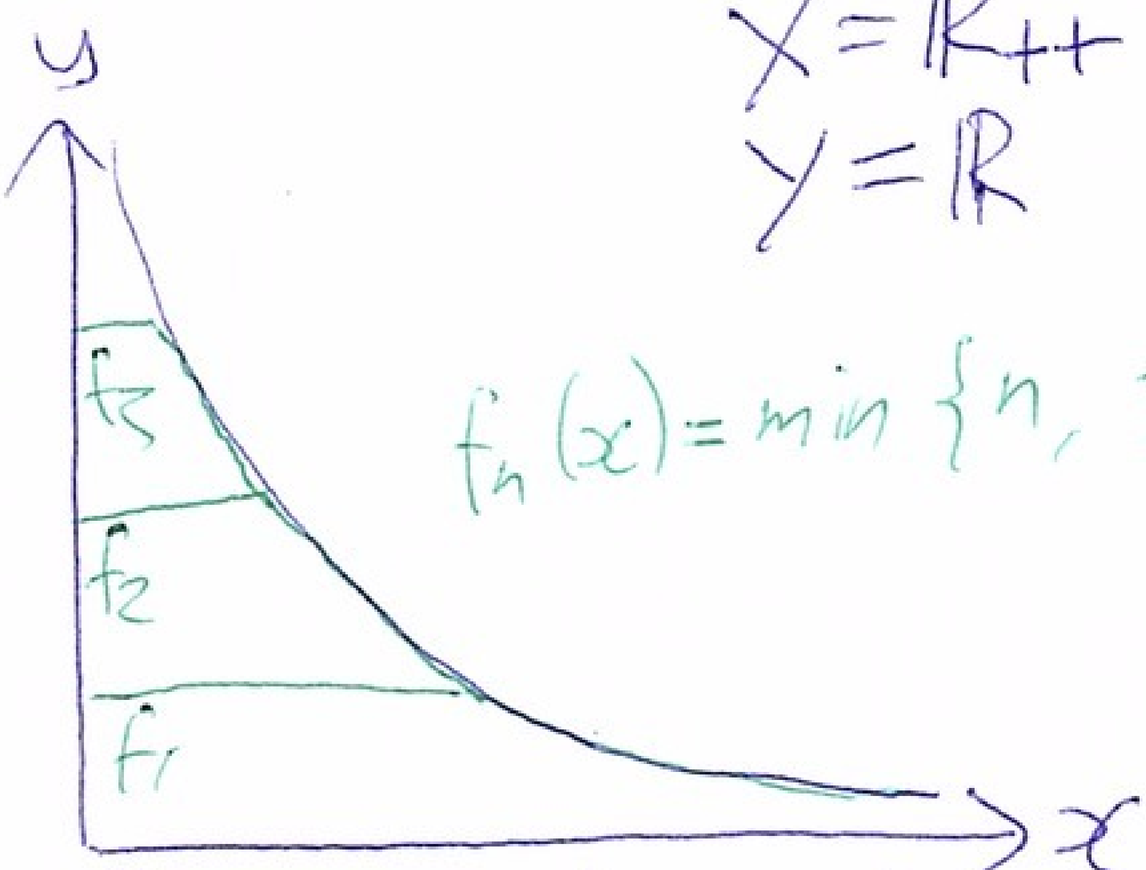
$$d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$



f_n is <u>not</u> a Cauchy sequence

$X = \mathbb{R}_{++}$
$Y = \mathbb{R}$

$f_n(x) = \min\{n, \frac{1}{x}\}$



f_n is <u>not</u> a Cauchy sequence.