From last time:

<u>Theorem</u> Let $A$ be a set in $(X, d)$. Then $A$ is closed $\Leftrightarrow$ $\partial A \subseteq A$.

<u>Proof</u> $\Rightarrow$ Suppose $x \in \partial A$. We need to prove that $x \in A$. By the first criterion of being a boundary point, there is some sequence $a_n \in A$ s.t. $a_n \to x$. ~~some~~ Since $A$ is closed, $x \in A$.

$\Leftarrow$ Suppose $\partial A \subseteq A$ and $a_n \in A$ and $a_n \to x$. We want to prove that $x \in A$. Suppose for the sake of contradiction that $x \notin A$. Let $b_n = x \in X \setminus A$, and $b_n \to x$. So $x \in \partial A \subseteq A$. $\lightning$

<u>Def</u> Let $A$ be a set in $(X, d)$. The <u>closure</u> of $A$ is

$$cl(A) = \overline{A} = \{x \in X : \text{there is a sequence } a_n \in A \text{ s.t. } a_n \to x\}.$$

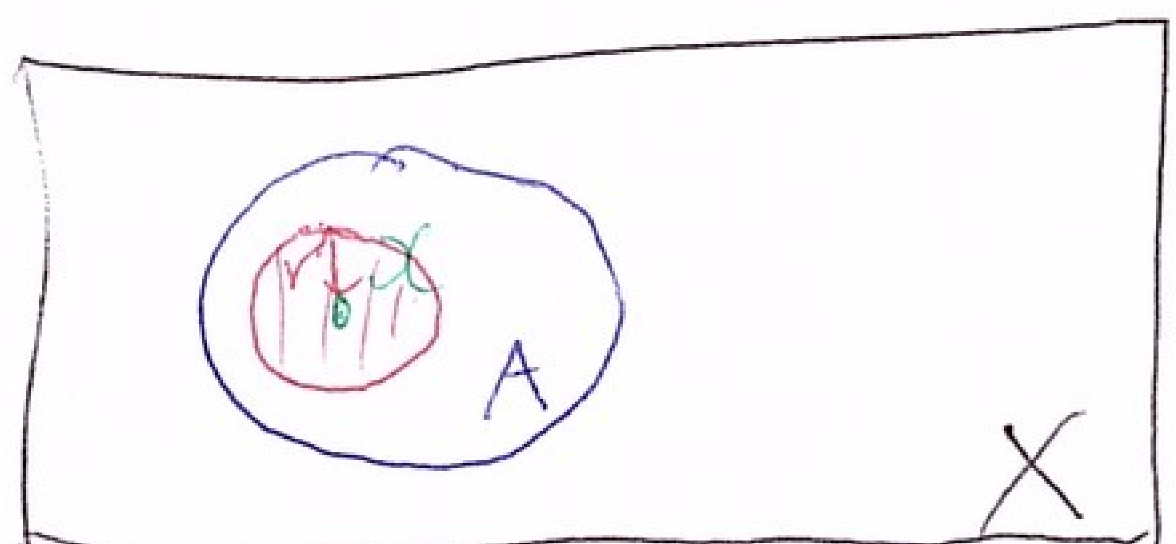
## C.5 Open sets

**Def** The *open ball* centred at $x$ with radius $r > 0$ in $(X, d)$ is

$$N_r(x) = \{y \in X : d(x,y) < r\}.$$

**Def** Let $A$ be a set in $(X, d)$.
We say $x \in A$ is an *interior point* if there is some $N_r(x)$ such that $N_r(x) \subseteq A$.

The set of interior points of $A$ is called *the interior of A*, or $\text{int}(A)$ or $A^{\circ}$.



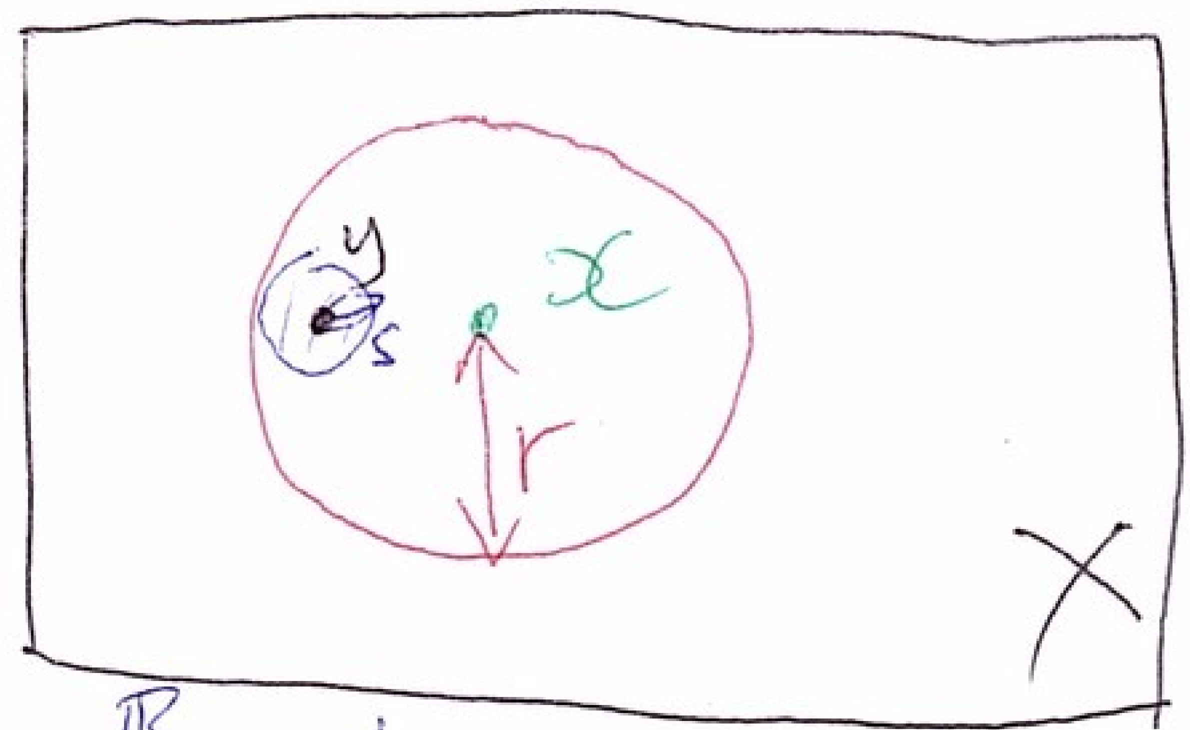

$x$ is an interior point of $A$.

**Def** Let $A$ be a set in $(X, d)$. We say $A$ is an *open set* if $A = \text{int}(A)$.

**Def** Suppose $A$ is an open set and $x \in A$. We say that $A$ is an *open neighbourhood* of $x$.

e.g. We say a utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$ is *locally non-satiated* if for every $x \in \mathbb{R}^N_+$, and every open neighbourhood $A$ of $x$ contains some $y \in A$ s.t. $u(y) > u(x)$.

eg: * $N_r(x)$ is an open set in $(X, d)$



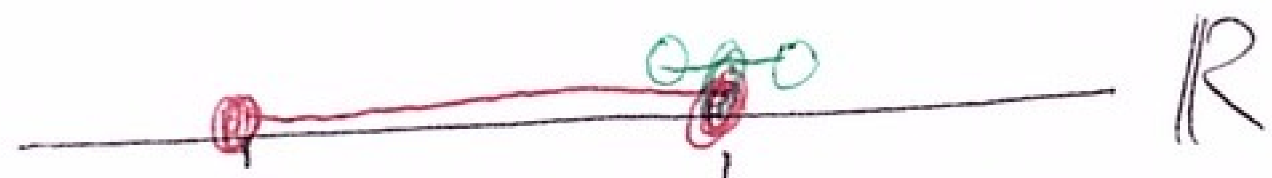

$s = r - d(x,y)$

$N_s(y) \subseteq N_r(x)$

* $(0,1)$ in $(\mathbb{R}, d_2)$ is open set, since $(0,1) = N_{\frac{1}{2}}(\frac{1}{2})$.

* In $(X, d)$, both $X$ and $\emptyset$ are open sets.

* $[0,1]$ is an open set in $([0,1], d_2)$ but NOT in $(\mathbb{R}, d_2)$.




$1 \notin \operatorname{int}([0,1])$

— $[0,1]$ is not open

Theorem Let $A$ be any set in $(X, d)$.
$A$ is open $\iff$ $A$ contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

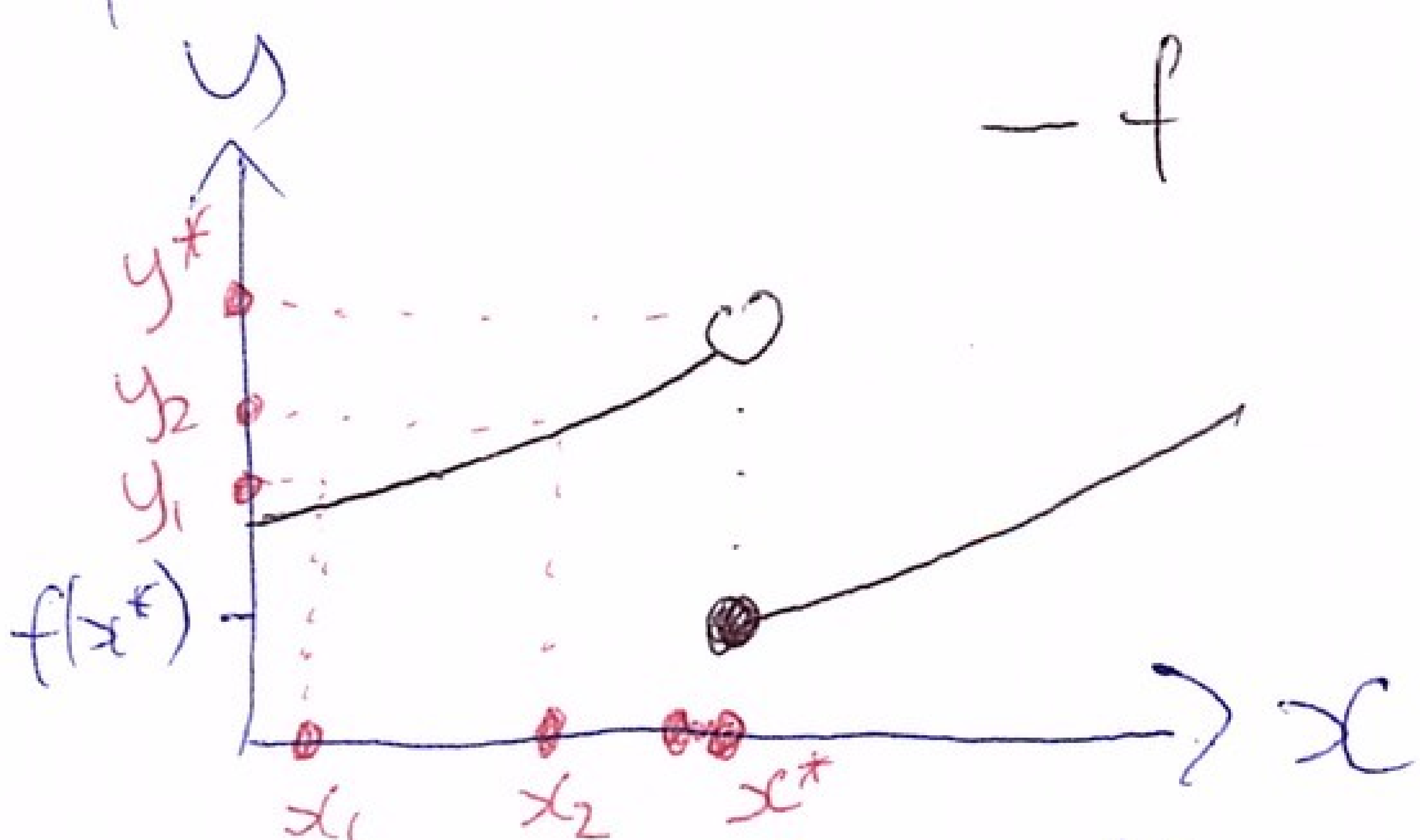
eg: $[0,1)$ in $(\mathbb{R}, d_2)$ is neither open nor closed, since $\partial[0,1) = \{0,1\}$ and not open ($0 \in [0,1)$) and not closed ($1 \notin [0,1)$).

eg: $[0,1]$ inside $([0,1] \cup [2,3], d_2)$ is both open and closed, since $\partial[0,1] = \emptyset$

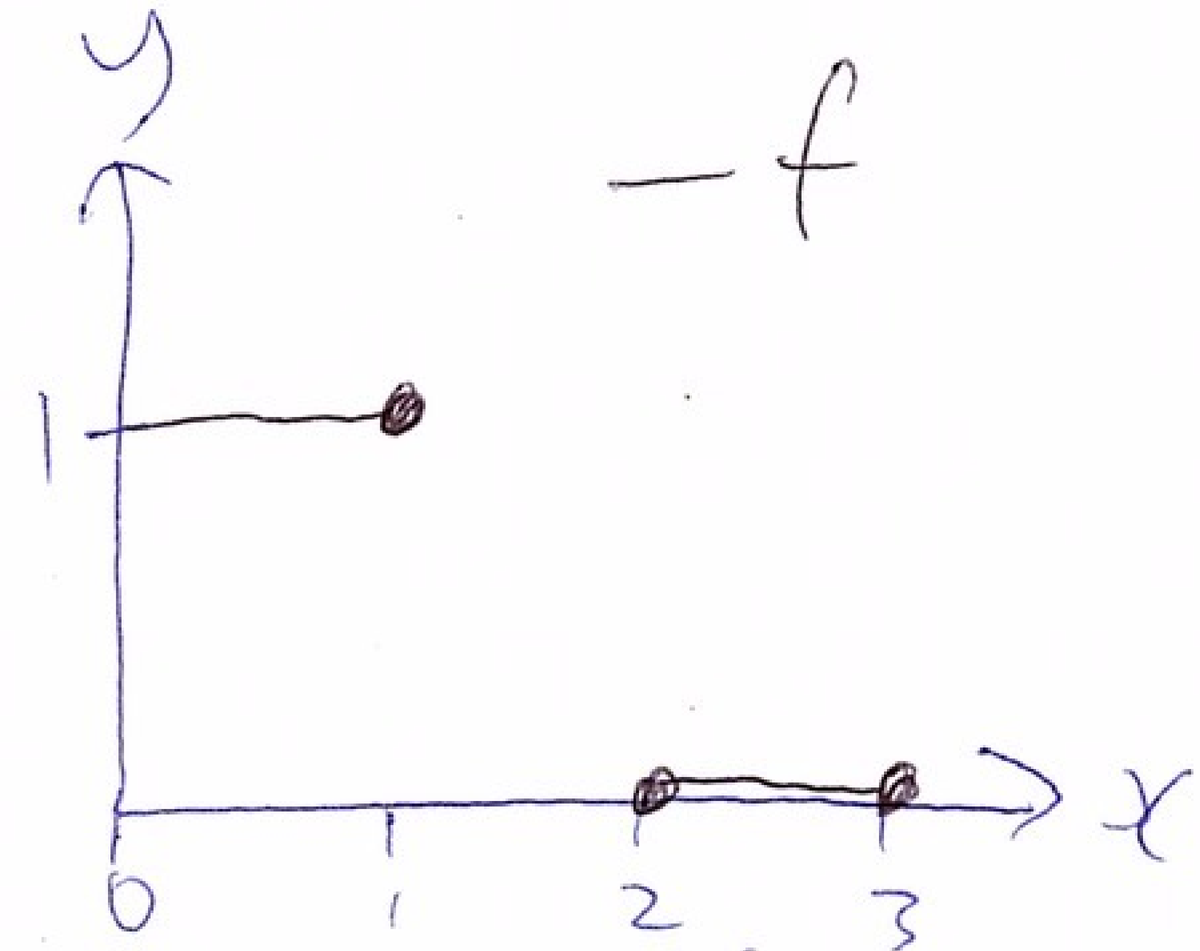
**Theorem** Let $A$ be any set in $(X,d)$. Then $A$ is open $\iff X \setminus A$ is closed.

## C.6 Continuity

**Def** Consider two metric spaces $(X, d_X)$ and $(Y, d_Y)$. We say that $f: X \to Y$ is <u>continuous at $x^* \in X$</u> if for every $x_n \in X$ with $x_n \to x^*$, the corresponding sequence $y_n = f(x_n)$ converges with $y_n \to f(x^*)$. If $f$ is continuous at all $x \in X$, then $f$ is <u>continuous</u>.




$f$ is not continuous
$y_n \to y^* \neq f(x^*)$




$f$ is continuous
$X = [0,1] \cup [2,3]$