Def Let x_n be a sequence in (X, d).
We say x_n is a **bounded sequence** if there exists some radius $r > 0$ such that

$$d(x_n, x_0) < r \text{ for all } n.$$

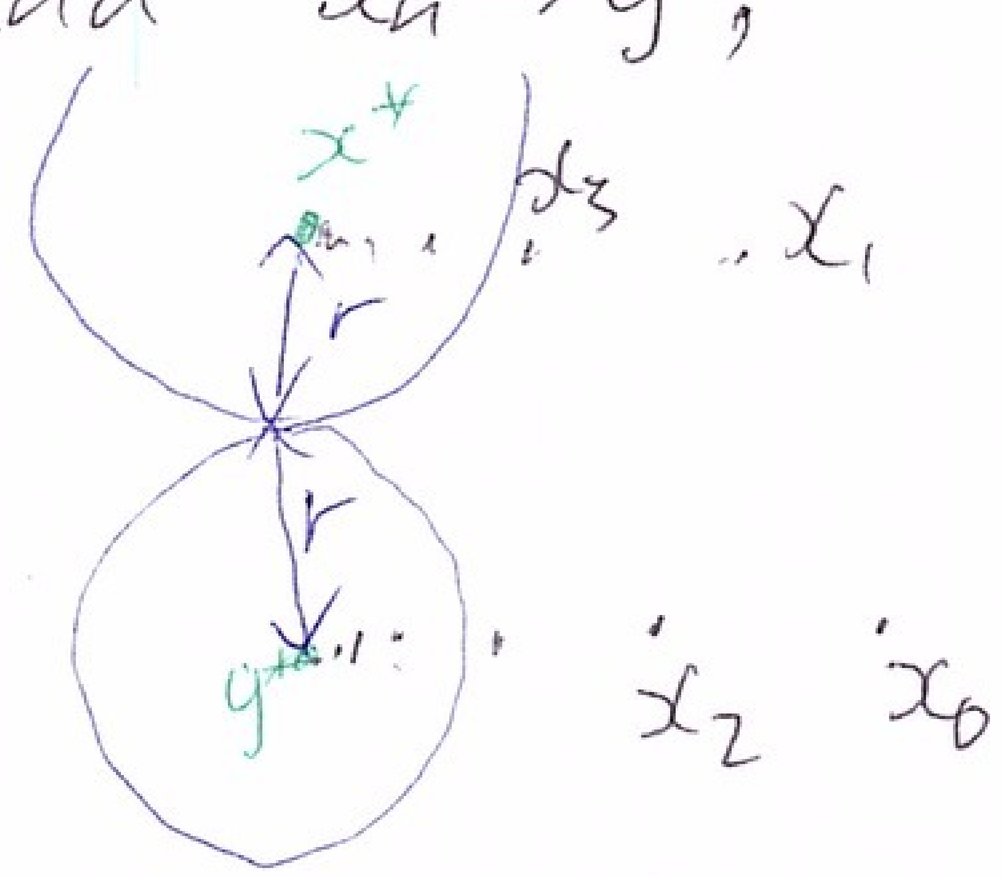
Otherwise we say x_n is an **unbounded sequence**.

Theorem Let x_n be a sequence in (X, d). If x_n is unbounded then x_n does not converge.

contrapositive: If x_n converges, then x_n is bounded.

Theorem A sequence x_n in (X, d) can converge to at most one point (in X).

Proof Suppose for the sake of contradiction that $x_n \to x^*$ and $x_n \to y^*$, and that $x^* \neq y^*$.



Let $r = \frac{1}{2} d(x^*, y^*)$. Since $x_n \to x^*$ and $x_n \to y^*$, there exists some N such that

$$d(x_n, x^*) < r \text{ and } d(x_n, y^*) < r$$

for all $n \geq N$.
That means $d(x_N, x^*) < r$ and $d(x_N, y^*) < r$.
So $d(x^*, y^*) \leq d(x^*, x_N) + d(x_N, y^*) < 2r = d(x^*, y^*)$. ↯

Def We say that y_n is a <u>subsequence</u> of x_n if there exists an increasing sequence $k_n \in \mathbb{N}$ (i.e with $k_{n+1} > k_n$) such that $y_n = x_{k_n}$.

eg: $x_n = n^2 = 0, 1, 4, 9, 16, 25, \dots$
$k_n = 2n$
$y_n = x_{k_n} = 0, 4, 16, \dots$

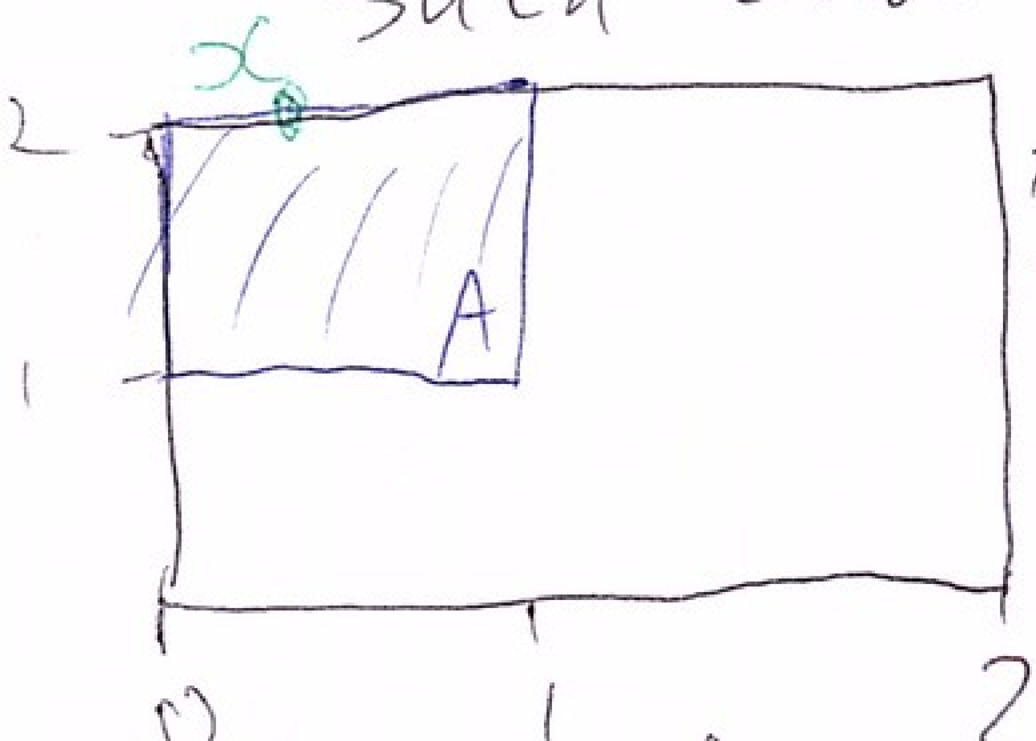
<u>Theorem</u>. If $x_n \to x^*$ and y_n is a subsequence of x_n, then $y_n \to x^*$.

C.3 Boundaries

Def Let A be any subset of (X, d). A point $x \in X$ is a <u>boundary point</u> of A if:

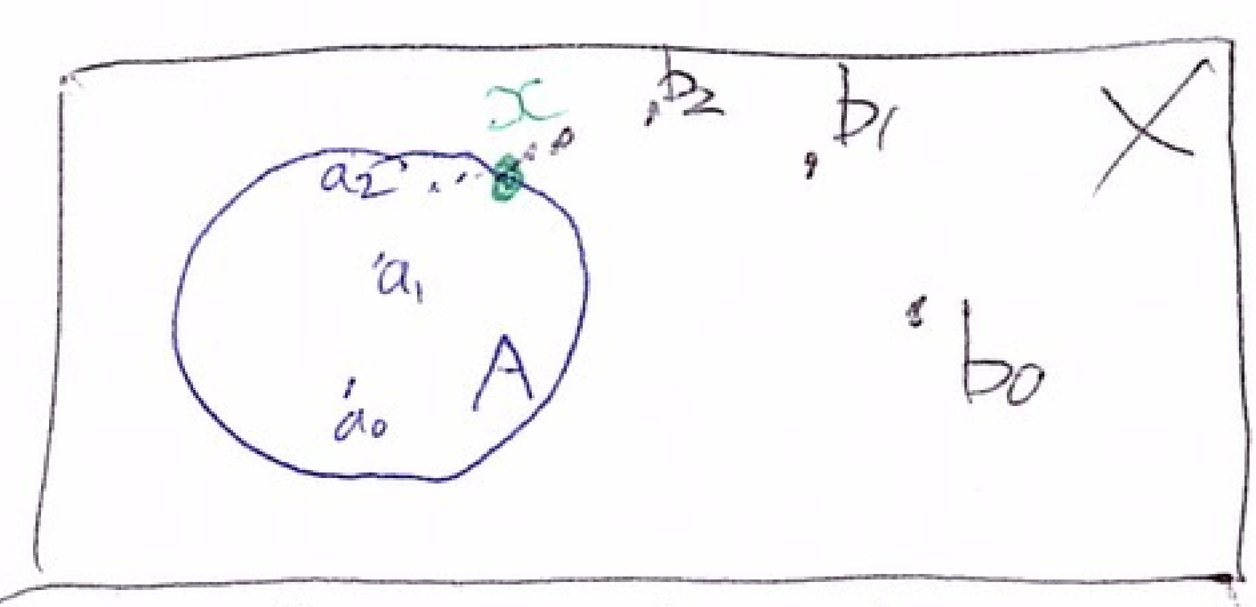
(i) there exists a sequence $a_n \in A$ such that $a_n \to x$, and

(ii) there exists a sequence $b_n \in X \setminus A$ such that $b_n \to x$.

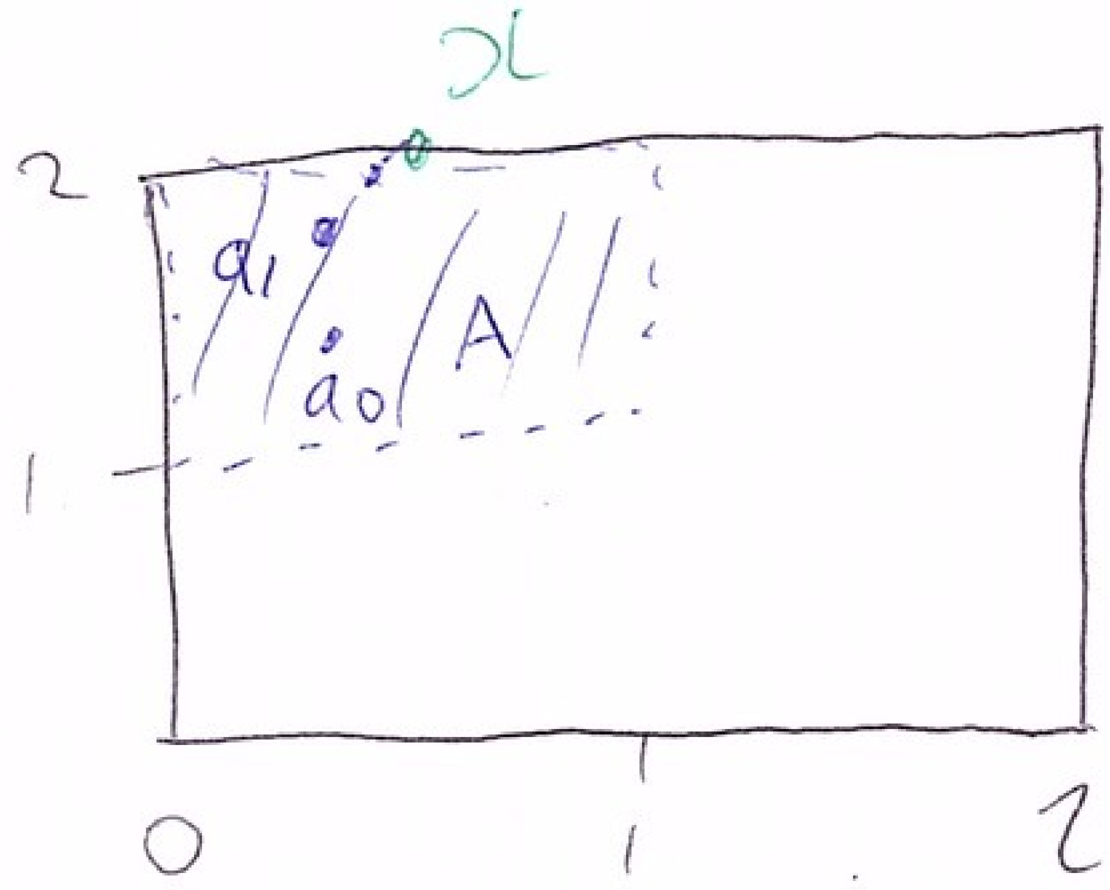


$A = [0,1] \times [1,2]$
$X = [0,2] \times [0,2]$

x fails the second criterion.



x is a boundary point of A.



$X = [0,2]\times[0,2]$

$A = (0,1)\times(1,2)$

x is a boundary point of A:

(i) We can find a sequence $a_n \in A$ such that $a_n \to x$, and [e.g. $a_n = \left(\frac{1}{2}, 2-\frac{1}{n+1}\right)$]

(ii) $b_n = x$ is a sequence $b_n \in X\setminus A$ s.t. $b_n \to x$.

Def The <u>boundary</u> of A, denoted ∂A is the set of boundary points of A.

C.4 Closed sets

Def Suppose A is a subset of (X,d). We say A is closed if there is no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$.

eg. $[0,1]$ is a closed set in $(\mathbb{R}, d_2)$

$(0,1)$ is NOT a closed set in $(\mathbb{R}, d_2)$ because $a_n = \frac{1}{n} \to 0 \notin (0,1)$.

X and $\emptyset$ in (X,d).

<u>Theorem</u> Suppose A is a set in (X, d). then A is closed if and only if A contains its boundary, i.e. $\partial A \subseteq A$.

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