$x, y \in \boxed{\mathbb{R}^n}$

$\boxed{d_2(x,y)} = \sqrt{\sum_{i=1}^{n} (x_i + y_i)^2}$ ← "Euclidean distance" (Pythagoras)

<u>Def (C.1)</u> (X, d) is a <u>metric space</u> consisting of a <u>point set</u> X and a <u>distance metric</u> $d : X \times X \to \mathbb{R}_+$ if:

(i) $d(x,y) = 0 \iff x = y$,

"if and only if"

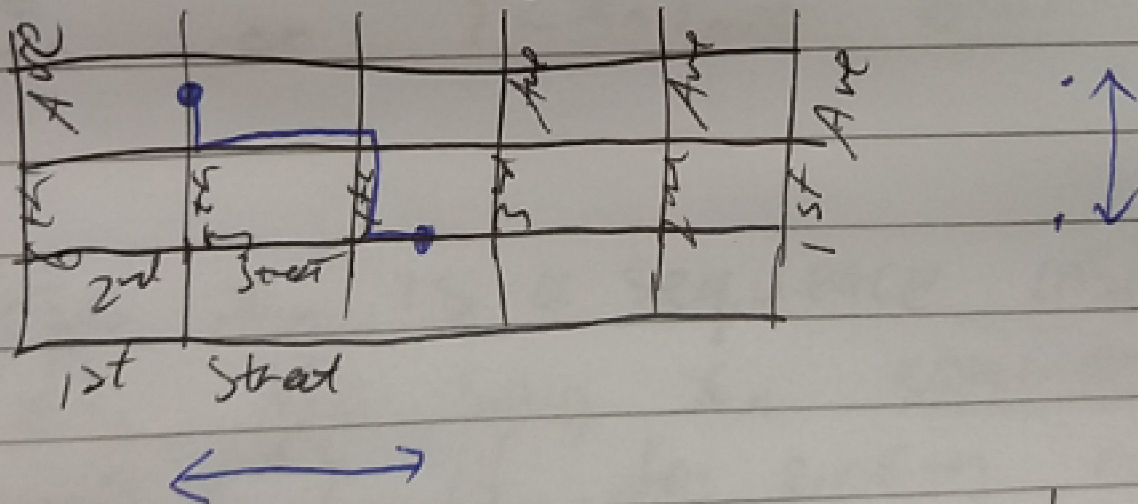
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$, and

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

"triangle inequality"
or
"no shortcuts property"

Examples:

* $(\mathbb{R}^n, d_2)$ Euclidean space
* $(\mathbb{R}^n, d_1)$, $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ ← Manhattan metric

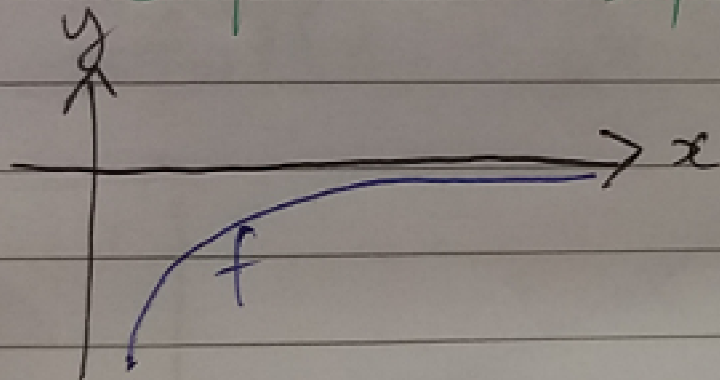


* If (X, d) is a metric space, and $Y \subseteq X$ then (Y, d) is a metric space.

* $(\mathbb{R}^n, d_\infty)$, $\max_i |x_i - y_i| = d_\infty(x, y)$

* (X, d_∞) where $X = \{f : [0,1] \to [0,1]\}$
and $d_\infty(f, g) = \sup\limits_{x \in X} |f(x) - g(x)|$.

sup ⟸ "supremum"

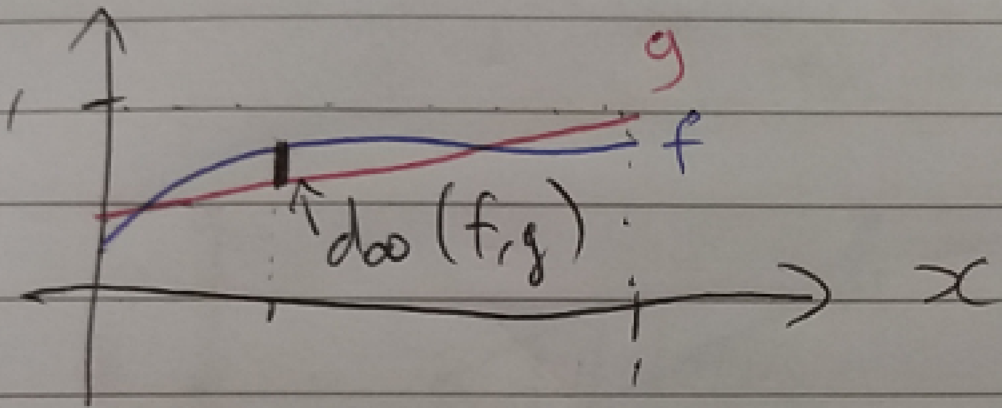


$f(x) = -\frac{1}{x}$

$f : \mathbb{R}_+ \to \mathbb{R}$

$\max\limits_{x \in \mathbb{R}_+} f(x)$ is <u>undefined</u>

$\sup\limits_{x \in \mathbb{R}_+} f(x) = 0.$



C.2 Sequences and convergence

Def A sequence in the set X is any function with domain $\mathbb{N}$ and co-domain X.

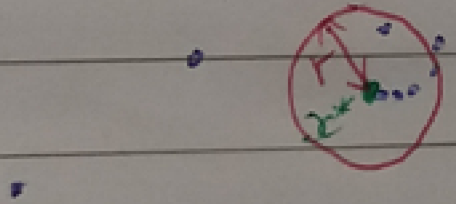
$x_0, x_1, x_2, \dots$

$\{x_n\}_{n=0}^{\infty}$ or $\{2x_n\}_{n=0}^{\infty}$ etc.

x_n

Def Suppose x_n is a sequence inside a metric space (X, d). We say x_n converges to $x^* \in X$ (write $x_n \to x^*$) if for every radius $r > 0$, there exists some $N \in \mathbb{N}$ such that

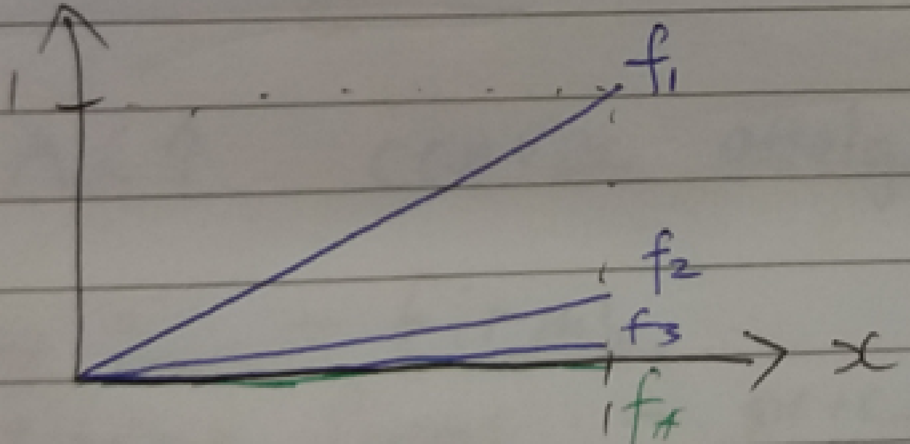
$$d(x_n, x^*) < r \quad \text{for every } n \geq N.$$



eg: $x_n = \frac{1}{n}$ inside $(\mathbb{R}, d_2)$ converges to 0.

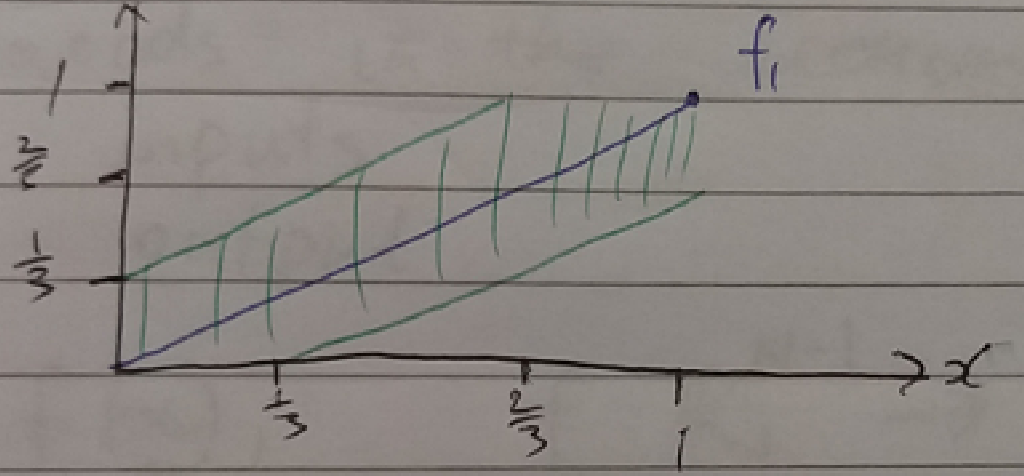
$x_n = \frac{1}{n}$ inside $(\mathbb{R}_{++}, d_2)$ does <u>not</u> converge.

$f_n(x) = \frac{x}{n^2}$ inside $(\{f: [0,1] \to [0,1]\}, d_\infty)$



$f_*(x) = 0$

$f_n \to f_*$


f_1
1
$\frac{2}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
1
x