Last time: we were proving

<u>Blackwell's Lemma</u> Suppose u is a bounded utility function, and $\beta < 1$.

Then the Bellman operator ("complicated" function)

$$F(\hat{V})(k) = \sup_{x, k' \geq 0} u(x) + \beta \hat{V}(k')$$
$$\text{s.t.} \quad x + k' = k$$

is a contraction of degree β on $(B(\mathbb{R}_+), d_\infty)$.

<u>Proof</u> We already proved $F: B(\mathbb{R}_+) \to B(\mathbb{R}_+)$.

We now prove F is a contraction.

Consider two ~~guesses~~ (candidate) value functions V_1 and V_2.

Let $x_1(k)$ and $x_2(k)$ be optimal policy functions for V_1 and V_2, respectively. $\boxed{!}$ Then,

$$F(V_1)(k) = u(x_1(k)) + \beta V_1(k - x_1(k))$$
$$= \left[u(x_1(k)) + \beta V_2(k - x_1(k)) \right] - \beta V_2(k - x_1(k)) + \beta V_1(k - x_1(k))$$
$$\leq \left[\max_{x \in [0,k]} u(x) + \beta V_2(k - x) \right] - \beta V_2(k - x_1(k)) + \beta V_1(k - x_1(k))$$

$$= u(x_2(k)) + \beta V_2(k - x_2(k))$$
$$- \beta V_2(k - x_1(k)) + \beta V_1(k - x_1(k))$$
$$= F(V_2)(k) - \beta V_2(k - x_1(k)) + \beta V_1(k - x_1(k))$$

Therefore,

$$F(V_1)(k) - F(V_2)(k) \leqslant \beta d_\infty(V_1, V_2).$$

Swapping the role of V_1 and V_2 gives

$$F(V_2)(k) - F(V_1)(k) \leq \beta d_\infty(V_1, V_2).$$

Combining, we get

$$|F(V_1)(k) - F(V_2)(k)| \leq \beta d_\infty(V_1, V_2)$$

for all V_1, V_2, k.

Therefore

$$d_\infty(F(V_1), F(V_2)) \leq \beta d_\infty(V_1, V_2).$$

So F is a contraction of degree β. $\square$

$$\max_{x \in [0,1)} x^2 \quad ?$$

There is no optimal choice!

Another version of Blackwell:

<u>Lemma</u>: Suppose u is bounded and continuous, and $\beta < 1$. Then the Bellman operator is a contraction of degree β on $(CB(\mathbb{R}_+), d_\infty)$.

<u>Proof</u>: We already established that F is a contraction on $(B(\mathbb{R}_+), d_\infty)$. It remains to show that if $V \in CB(\mathbb{R}_+)$, then $F(V)$ is a continuous function.

$k \longmapsto F(V)(k)$

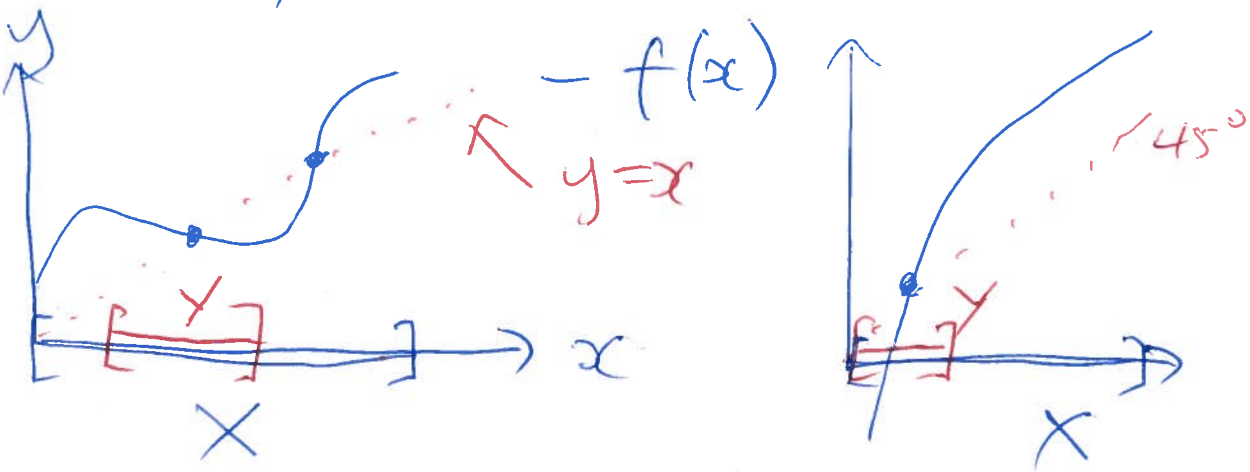
Skip rest of proof. $\square$

Using these tools:

* There is only one solution to the Bellman equation (cf. Banach's fixed point theorem).

That means there are no ~~only~~ "wrong" solutions among $(B(\mathbb{R}_+), d_\infty)$.

* We can learn properties of the value function — increasing, continuous, concave?



We learn the fixed point is in Y.

* We can get an algorithm.

C9 Compact Sets

<u>Def</u> Let A be a subset of (X, d). We say A is <u>compact</u> if every sequence $x_n \in A$ has a convergent subsequence $y_n \to y^*$ where $y^* \in A$.

We say (X, d) is a compact metric space if X is a compact set inside (X, d).

<u>Def</u> A set in (X, d) is <u>bounded</u> if it is contained in an open ball.

Theorem (Bolzano-Weierstrass)

Let A be a subset $(\mathbb{R}^n, d_2)$. Then A is compact set if and only if A is closed and bounded.

Proof

Suppose A is compact.

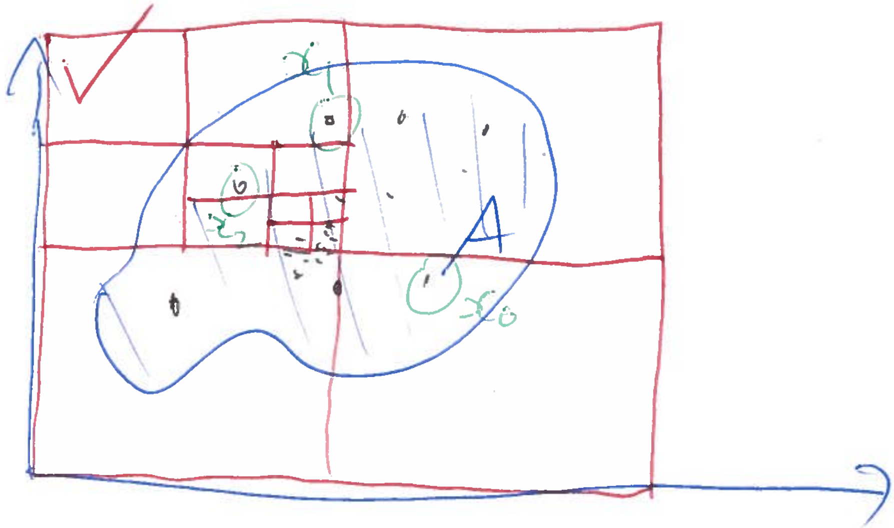
$\Rightarrow$ A is bounded. Suppose A were not bounded. Then there is a sequence x_n with the property $d(x_n, 0) > n$. Now x_n

$$\left.\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}\right\}n$$

is an unbounded, and so are all of its subsequences. So all subsequences do not converge. This contradicts A being compact.

$\Rightarrow$ A is closed: Suppose $y_n \in A$ is convergent with $y_n \to y^*$. Every subsequence of y_n also converges to y^*. By the definition of compactness, $y^* \in A$. Therefore A is closed.

closed + bounded $\Rightarrow$ compact:



We pick a x_0 out of the biggest box, x_1 out of a smaller box (top-left), x_2 out of a smaller box again.

<u>claim</u>: x_n is a subsequence that is a Cauchy sequence.

Since $(\mathbb{R}^n, d_2)$ are complete, x_n is convergent. $\square$