Appendix G cont'd

$$V_t(k) = \overset{\text{sup}}{\max_{\{x_s\}_{s=t}^{\infty}}} \sum_{s=t}^{\infty} \beta^{s-t} u(x_s)$$

$$\text{s.t. } \sum_{s=t}^{\infty} x_s = k.$$

Q: Is there an optimal solution?

New Bellman equation:

$$V_t(\underset{k}{k_t}) = \sup_{\overset{x}{x_t},\ \overset{k'}{k_{t+1}} \geq 0} u(\underset{x}{x_t}) + \beta V_{t+1}(\underset{k'}{k_{t+1}})$$

$$\text{s.t. } \overset{x}{x_t} + \overset{k'}{k_{t+1}} = \overset{k}{k_t}.$$

Time t is the "same" as $t+1$.

$V_0 = V_1 = V_2 = \cdots = V$.

Rewrite without t

$$V(k) = \sup_{x,\ k' \geq 0} u(x) + \beta V(k')$$

$$\text{s.t. } x + k' = k.$$

This is an equation with an unknown variable V appearing on both sides.

Similar to solving: $x = \sqrt{x} + x^2$?

C8 Fixed Points

Thinking about equations like

$$x = f(x).$$

<u>Def</u> A function f is a <u>self-map</u> if $f: X \to X$, i.e domain = co-domain

<u>Def</u> Let $f: X \to X$. We say $x^* \in X$ is a <u>fixed point</u> of f if $x^* = f(x^*)$.

<u>Def</u> Let (X, d_X) and (Y, d_Y) be metric spaces, and $a > 0$. A function $f: X \to Y$ is <u>Lipschitz continuous</u> of degree a if for every $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq a\, d_X(x, x').$$

HW: (Q4a) implies f is continuous.

Def Let (X,d) be a metric space. The self-map $f: X \to X$ is a contraction if it is Lipschitz continuous of degree $a<1$, i.e. $d(f(x), f(y)) \leq a\, d(x,y)$, for all $x, y \in X$.

~~Th~~ **Banach's Fixed Point Theorem**

Let (X,d) be a ~~an~~ complete metric space. If $f: X \to X$ is a contraction of degree a, then

(i) f has a unique fixed point x^*,
("the" fixed point)

"initial guess"
(ii) Given any $x_0 \in X$, the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*,

(iii) $d(x_n, x^*) \leq \dfrac{a^n}{1-a} d(x_0, x_1)$.

Proof Uniqueness: Suppose for the sake of contradiction that $x^*, x^{**} \in X$ ~~were~~ are fixed points of f, and $x^* \neq x^{**}$.

Being fixed points,

$$d(\underbrace{f(x^*)}_{x^*}, \underbrace{f(x^{**})}_{x^{**}}) = d(x^*, x^{**}).$$

But the contraction property requires

$$d(f(x^*), f(x^{**})) \le a\, d(x^*, x^{**}) < d(x^*, x^{**}).$$

A contradiction.

<u>Existence and convergence:</u>

Our main task is to prove that x_n is a Cauchy sequence.

$$d(x_1, x_2) = d(f(x_0), f(x_1)),$$

because $x_{n+1} = f(x_n)$. ← apply f n times

$$d(x_n, x_{n+m}) = d(f^{n}(x_0), f^n(x_m)) \quad \underbrace{f(f(f(\cdots}_{n}$$

Now, the contraction property implies

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \le a\, d(x_0, x_1)$$

$$d(x_n, x_{n+1}) = d(f^n(x_0), f^n(x_1)) = d(f(f^{n-1}(x_0)), f(f^{n-1}(x_1)))$$

$$\le a\, d(f^{n-1}(x_0), f^{n-1}(x_1))$$

$$\le a^2\, d(f^{n-2}(x_0), f^{n-2}(x_1))$$

$$\le a^n\, d(x_0, x_1)$$

$$\boxed{d(x_n, x_{n+m}) \le a^n d(x_0, x_m).}$$

This implies

$$d(x_0, x_m) \le \overbrace{d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m)}^{\text{triangle inequality}}$$

$$\le d(x_0, x_1) + d(x_1, x_2) + \cdots \quad \text{never ends}$$

$$\le d(x_0, x_1) \overset{\text{formula}}{+} a\, d(x_0, x_1) + a^2 d(x_0, x_1) + \cdots$$

$$= d(x_0, x_1) [\underbrace{1 + a + a^2 + \cdots}_{\text{geometric series}}]$$

$$= \frac{d(x_0, x_1)}{1-a}.$$

Combining, we get

$$d(x_n, x_{n+m}) \le \frac{a^n}{1-a} d(x_0, x_1).$$

Fix any radius $r > 0$, and let N be a number satisfying

$$\frac{a^N}{1-a} d(x_0, x_1) < \frac{r}{2}.$$

Then

$$d(x_n, x_m) \le d(x_n, x_N) + d(x_N, x_m)$$

$$\le \frac{a^N}{1-a} d(x_0, x_1) + \frac{a^N}{1-a} d(x_0, x_1)$$

$$< \frac{r}{2} + \frac{r}{2}$$

$$= r \quad \text{for all } n, m \ge N.$$

So x_n is a Cauchy sequence.

Since (X, d) is a complete metric space, and x_n is a Cauchy sequence, x_n converges to some point $x^* \in X$, i.e. $x_n \to x^*$.

By continuity of f, $\underbrace{f(x_n)}_{y_n} \to f(x^*)$.

Since $y_n = f(x_n) = x_{n+1}$ is a subsequence of x_n. So $y_n \to x^*$.

Since $y_n \to x^*$ and $y_n \to f(x^*)$ we conclude $x^* = f(x^*)$. So x^* is the fixed point.

<u>Approximation bound:</u> ~~By cont~~

$$
\begin{aligned}
d(x_n, x^*) &= \lim_{m \to \infty} d(x_n, x_m) && \text{Since } d \text{ is continuous and } x_m \to x^* \\
&\leq \lim_{m \to \infty} \frac{a^n}{1-a} d(x_0, x_1) && \text{by formula} \\
&= \frac{a^n}{1-a} d(x_0, x_1). && \square
\end{aligned}
$$

Appendix G (again)

The Bellman operator is

$$F(V')(k) = \sup_{x,k' \geq 0} u(x) + \beta V'(k')$$

tomorrow's value function

$$\text{s.t. } x + k' = k,$$

which is a function whose domain and co-domain is a set of possible value functions, e.g. $(B(\mathbb{R}), d_\infty)$.

$$F(V') = \left[k \mapsto \begin{array}{l} \sup_{x,k' \geq 0} u(x) + \beta V'(k') \\ \text{s.t. } x + k' = k \end{array} \right]$$

The Bellman equation becomes

$$V = F(V).$$

Remaining task: prove F is a contraction.

Blackwell's Lemma (1965)

Suppose u is a bounded utility function, i.e. $u \in B(\mathbb{R}_+)$.
Then the Bellman operator is a contraction of degree β on $(B(\mathbb{R}_+), d_\infty)$.

Proof ~~Fix any~~ F is a self-map:

Fix any $V' \in B(\mathbb{R}_+)$. We first show $F(V')$ exists and $F(V')$ is bounded, i.e. $F(V') \in B(\mathbb{R}_+)$. Since u and V' are bounded, there open balls $N_r(0)$ and $N_s(0)$ containing the ranges of u and V' respectively. Therefore every choice (x, k') involves the objective lying inside $N_{r+\beta s}(0)$. So the supremum exists and $F(V')$ exists and it is bounded.