C.4 Closed sets

<u>Def</u> Let A be any set in a metric space (X, d). The <u>closure of A</u> is

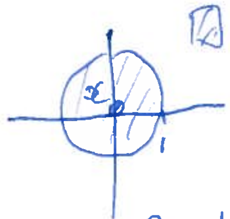
$$cl(A) = \bar{A} = \{x^* \in X : \text{there exists a sequence } x_n \in A \text{ with } x_n \to x^*\}.$$

C.5 Open sets

<u>Def</u> The <u>open ball</u> centred at $x \in X$ with radius $r > 0$ in the metric space (X, d) is

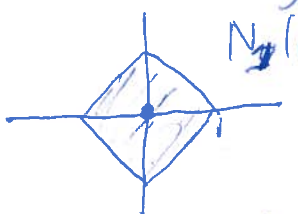
$$N_r(x) = \{y \in X : d(x, y) < r\}.$$

$N_1(0)$



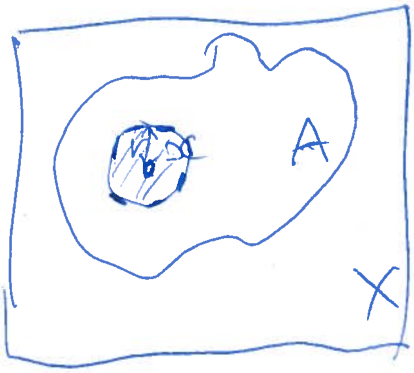
In $(\mathbb{R}^2, d_2)$

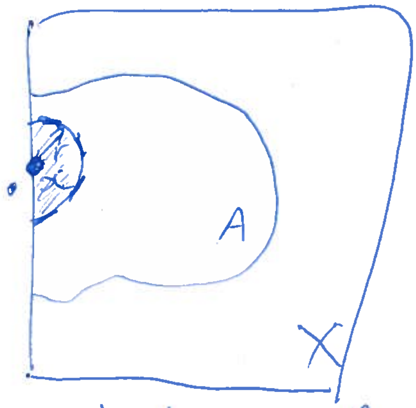
$N_1(0)$



In $(\mathbb{R}^2, d_1)$

<u>Def</u> Suppose A is a subset of a metric space (X, d). We say that a point $x \in A$ is an <u>interior point</u> of A if there exists an open ball $N_r(x)$ such that $N_r(x) \subseteq A$.



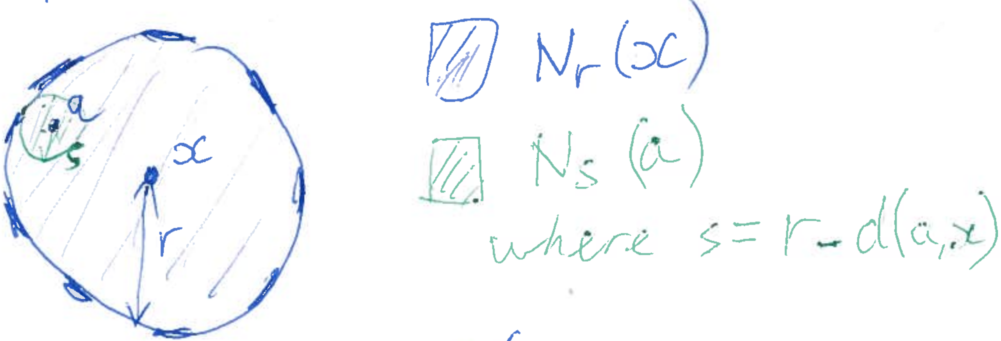


The set of interior points of A is called the <u>interior of A</u>.

We say that A is an <u>open set</u> if A equals its interior, i.e. every $x \in A$ is an interior point of A. If A is an open set and $x \in A$, then we say A is an <u>open neighbourhood</u> of x.

(Non)-examples

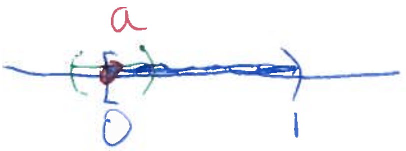
* $N_r(x)$ is open in (X, d)



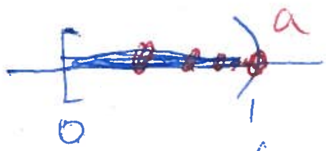
Open balls are open sets.

* $(0,1)$ is open in $(\mathbb{R}, d_2)$.
Special case of the previous example, because $(0,1) = N_{\frac{1}{2}}(\frac{1}{2})$.

* In (X, d), both $\emptyset$ and X are open sets.
  Recall $\emptyset$ and X are also closed sets.
* $[0,1)$ in $(\mathbb{R}, d_2)$ is neither open nor closed.



not open ~~in $(\mathbb{R}, d_2)$~~ because $(\;)$ is not contained within $[0,1)$



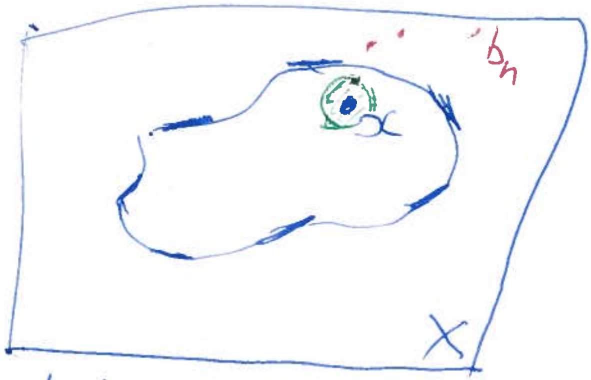
not closed because $a_n = 1 - \frac{1}{n+2} \to 1$, and $1 \notin [0,1)$.

* $[0,1]$ is an open set in $([0,1], d_2)$ but <u>NOT</u> in $(\mathbb{R}, d_2)$.

<u>Theorem</u> Let A be a set in a metric space (X, d). Then A is open if and only if A contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

<u>Proof</u> <u>open $\Rightarrow$ contains none of boundary:</u>

Consider any point $x \in A$. We need to show $x \notin \partial A$. Since A is open, we can find an open ball $N_r(x)$ such that $N_r(x) \subseteq A$. Thus it is impossible to find a sequence $b_n \notin A$ and $b_n \to x$. We conclude that x is not a boundary point.

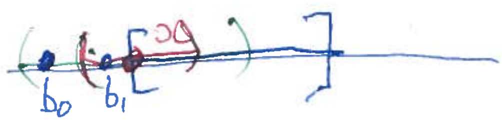


not ~~is~~ open $\Rightarrow$ contains some of boundary

(instead of "contains none of boundary $\Rightarrow$ open")

Suppose A is not open. There must be some point $x \in A$ such that every $N_r(x) \not\subseteq A$. Let $r_n = \frac{1}{n+1}$.

For every n, there is a point $b_n \in N_{r_n}(x)$ but $b_n \notin A$.



Since $d(x, b_n) < r_n$, we conclude that $b_n \to x$.

Moreover, the trivial ~~seq~~ sequence $a_n = x$ also has $a_n \to x$.

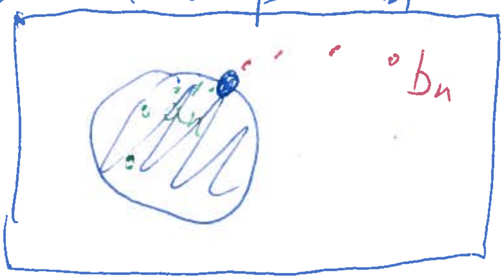
Therefore, $x \in \partial A$. $\square$

~~More~~ Another example:

* $[0,1]$ is open (and closed) inside $([0,1] \cup [2,3], d_2)$, because $\partial[0,1] = \emptyset$.

Theorem Let A be a subset of a metric space (X, d). Then A is open if and only if $X \setminus A$ is closed.

Proof Trick: $\partial A = \partial(X \setminus A)$
— just swap a_n and b_n.



If A is open, it contains none of its boundary, so $X \setminus A$ contains all its boundary. So ~~A~~ $X \setminus A$ is closed. etc. $\square$

Back to 2.1 Production functions

* decreasing marginal productivity

$\dfrac{\partial f(x)}{\partial x_1}$ is decreasing as x_1 increases

(and $x_2, \dots, x_{N-1}$ stay the same).

* weakly increasing returns to scale:

for all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$,

$$f(tx) \geq t f(x)$$

e.g. $t = 2$ $$\underbrace{f(2x)}_{\text{reorganise a single factory}} \geq \underbrace{2f(x)}_{\text{2 identical factories}}$$

* constant returns to scale:

for all $x \in \mathbb{R}_+^{N-1}$ and all $t > 0$,

$$f(tx) = t f(x).$$

Common assumption.

* weakly decreasing returns to scale:

for all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$,

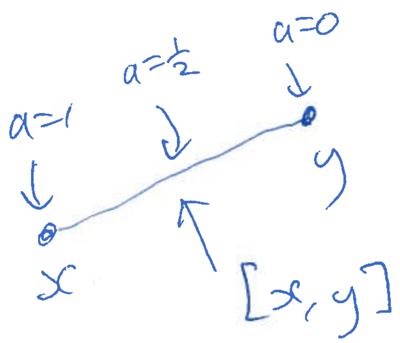
$$f(tx) \leq t f(x).$$

Goes against the philosophy of G.E!

~~Ap~~ D – Convex Geometry

Def A <u>closed interval</u> between two points $x, y \in \mathbb{R}^n$ is

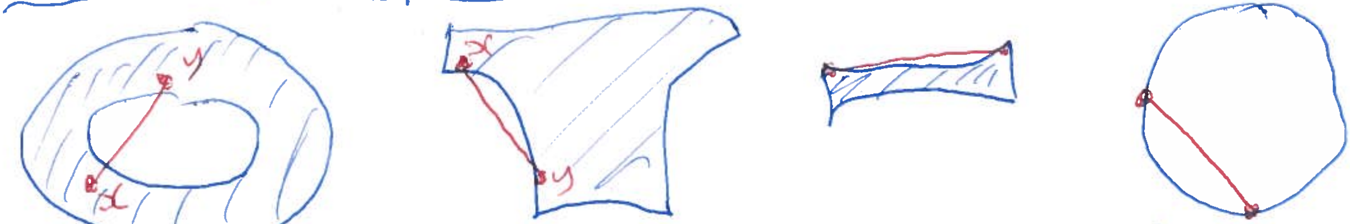
$$[x,y] = \{\underbrace{ax + (1-a)y}_{\text{convex combination}} : a \in [0,1]\}$$



$[x,y)$, (x,y), etc are similar.

Def $X \subseteq \mathbb{R}^n$ is a <u>convex set</u> if for all $x, y \in X$, the interval $[x,y] \subseteq X$.

Non-examples



skinny donut

Examples

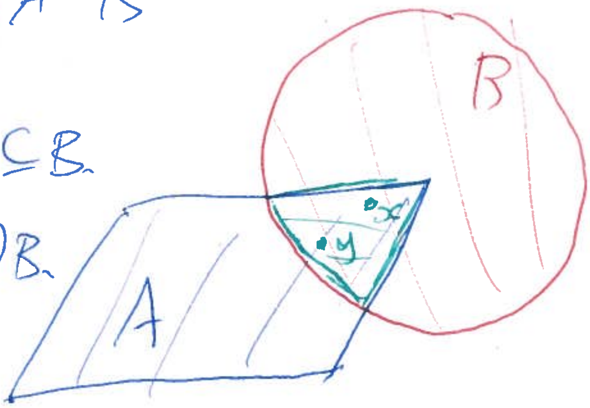
Theorem If A and B are convex sets, then $A \cap B$ is a convex set.

Proof Suppose A and B are convex. We want to show $A \cap B$ is convex. Pick any $x, y \in A \cap B$. We want to show $[x,y] \subseteq A \cap B$.

First, we show $[x,y] \subseteq A$. Since $A \cap B \subseteq A$, we know $x \in A$ and $y \in A$. Since ~~[x,y]~~ A is convex, $[x,y] \subseteq A$.

Similarly, $[x,y] \subseteq B$.

Therefore $[x,y] \subseteq A \cap B$.



$\square$

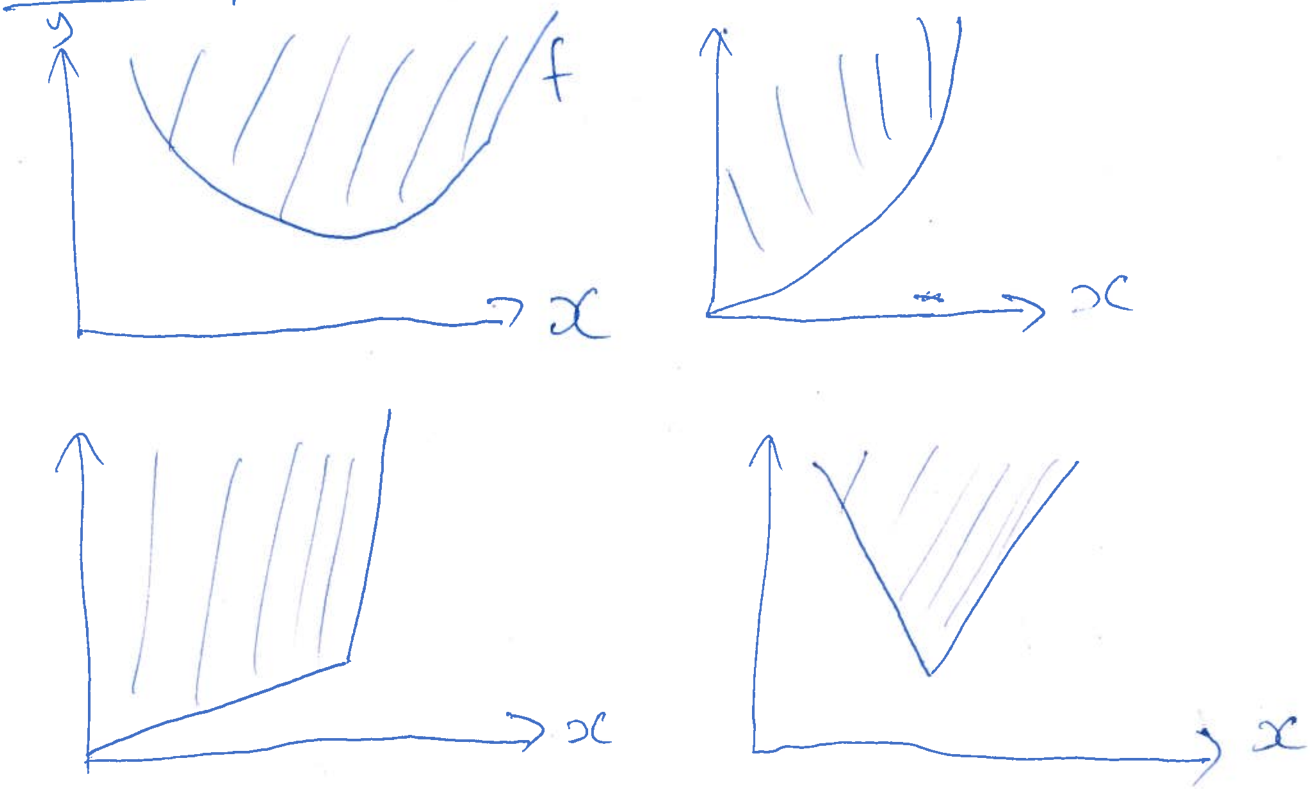
Def Let $X \subseteq \mathbb{R}^n$. $f: X \to \mathbb{R}$ is a <u>convex function</u> if its <u>hypergraph</u>

$$\text{hyper}(f) = \{(x,y) : x \in X, y \geq f(x)\}$$

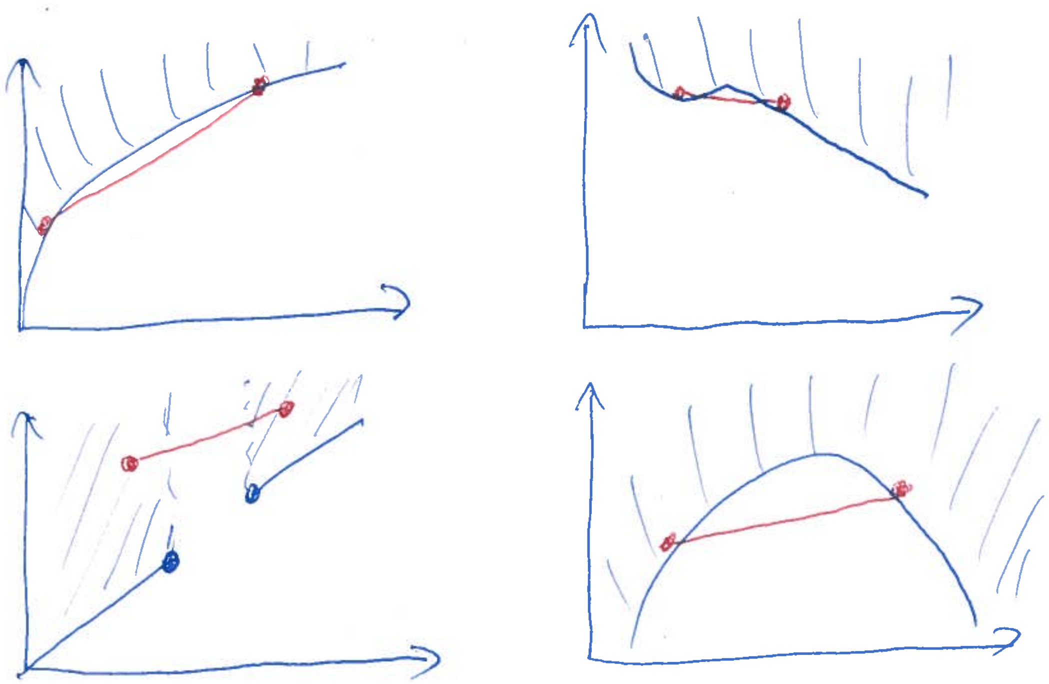
is a convex set.

Examples:

$\boxed{/\!/}$ hyper(f)



Non-examples:



Def $f: X \to \mathbb{R}$ is a concave function if its ~~hypogr~~ hypograph

$$\mathrm{hypo}(f) = \{(x,y) : x \in \mathbb{R},\ y \le f(x)\}$$

is a convex set.

Note: there is no such thing as a concave set.

concave | concave & convex | neither concave nor convex

Theorem D2 If $f: X \to \mathbb{R}$ is a convex function ~~then~~ and $X \subseteq \mathbb{R}^n$ is an open set in $(\mathbb{R}^n, d_2)$, then f is continuous.

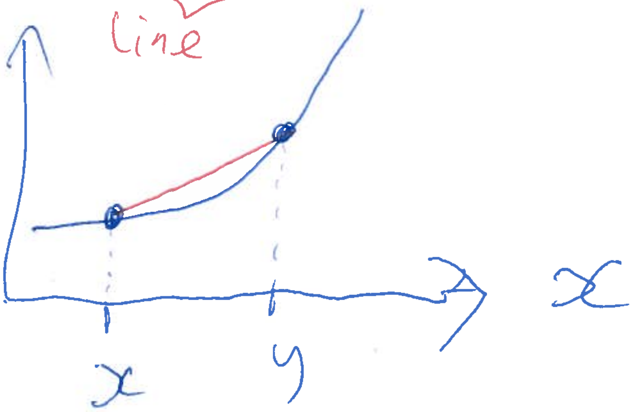
Theorem D3 Suppose $f: \mathbb{R} \to \mathbb{R}$ is ~~a~~ differentiable. Then f is convex if and only if its derivative f' is weakly increasing.

Theorem D4 Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is convex if and only if $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

Theorem D6 ~~Suppose $f: X \to \mathbb{R}$~~

A function $f: X \to \mathbb{R}$ is a convex function if and only if for all $x, y \in X$ and all $a \in (0,1)$ weights

$$\underbrace{af(x) + (1-a)f(y)}_{\text{line}} \geq \underbrace{f(ax + (1-a)y)}_{\text{curve}}.$$



Back to 2.1:

* concavity: f is a concave function. Connections:
  - if f is smooth and concave, then f has decreasing marginal productivity

- if f is concave and $f(0)=0$, then it has (weakly) decreasing returns to scale.

<u>Proof</u> We must show $f(tx) \leq t f(x)$ for all $x \in \mathbb{R}_+^{N-1}$ and $t > 1$.

Let $A = \frac{1}{t}$, $Y = 0$, $X = tx$.

Applying theorem D6, f is concave iff:

$$A f(X) + (1-A) f(0) \leq f(AX + (1-A)0)$$

$$\iff \frac{1}{t} f(tx) + \left(1 - \frac{1}{t}\right) 0 \leq f\left(\frac{1}{t} tx + 0\right)$$

$$\iff \frac{1}{t} f(tx) \leq f(x)$$

$$\iff f(tx) \leq t f(x). \quad \square$$