C2 Convergence

$= \{0, 1, 2, \dots\}$

Def A sequence in the set X is a function with domain $\mathbb{N}$ and co-domain X. Sequences have special notation:

$$x_0, \ x_1, \ x_2, \ \dots$$

$$\{x_n\}_{n=0}^{\infty}$$

$$x_n$$

Note: sequences are infinite.

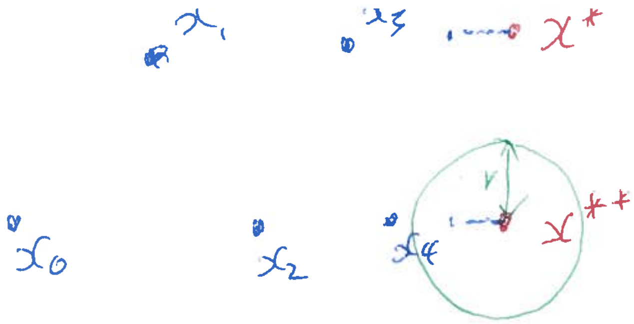
Def Suppose x_n is a sequence inside a metric space (X, d). We say that x_n converges to $x^* \in X$ (or write $x_n \to x^*$) if for every radius $r > 0$ there exists some N such that

$$d(x_n, x^*) < r \quad \text{for all } n \geq N.$$

x^* is called the limit of x_n.

$N = 4$

$x_n \to x^*$



$x_n \not\to x^{**}$

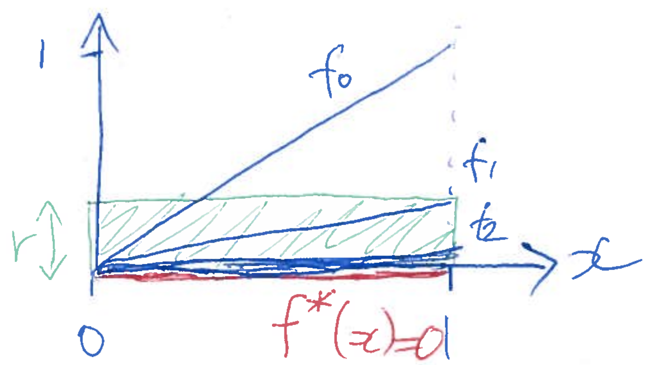
x_n is <u>non-convergent</u>

$x_n = \frac{1}{n+1}$ inside $(\mathbb{R}, d_2)$

<u>claim</u>: $x_n \to 0$

$(C([0,1],[0,1]), d_\infty)$

Any $g \in \boxed{}$ has $d_\infty(g, f^*) < r$



$f_n \to f^*$

where $f_n(x) = \frac{x}{n}$.

<u>claim</u> $f_n \to f^*$

<u>Proof</u>: $d_\infty(f_n, f^*) = \underbrace{f_n(1)}_{\frac{1}{n}} - \underbrace{f^*(1)}_{0} = \frac{1}{n}$.

Given a radius $r > 0$, if we pick $N = \frac{1}{r} + 1$, then $d_\infty(f_n, f^*) = \frac{1}{n} < r$ for all $n \geq N$. □

Algebra: if $n = N$, then $\frac{1}{n} = \frac{1}{\frac{1}{r}+1} = \frac{r}{1+r} < r$

More examples:

* $x_n = \frac{1}{n}$ does not converge in $(\mathbb{R}_{++}, d_2)$

$= \{x \in \mathbb{R} : x > 0\}$

Def Let x_n be a sequence in (X,d). We say x_n is **bounded** if there exists some radius $r>0$ such that $d(x_0, x_n) < r$ for all n. Otherwise x_n is **unbounded**.

Theorem If x_n is an unbounded sequence in (X,d), then x_n is not convergent.

Theorem A sequence x_n in (X,d) can converge to at most one point.

Proof: Suppose for the sake of contradiction that $x_n \to x^*$ and $x_n \to y^*$ and $x^* \neq y^*$.

Let $r = \frac{1}{2} d(x^*, y^*)$. Since $x_n \to x^*$, there is some N_x such that

$$d(x_n, x^*) < r \quad \text{for all } n \geq N_x.$$

Similarly, there some N_y such that

$$d(x_n, y^*) < r \quad \text{for all } n \geq N_y.$$

Let $N = \max\{N_x, N_y\}$. Then

$$d(x_n, x^*) < r \text{ and } d(x_n, y^*) < r \text{ for all } n \geq N.$$

By the triangle inequality,

$$
\begin{aligned}
d(x^*, y^*) &\le d(x^*, x_N) + d(y^*, x_N) \\
&\le \ \leftarrow \text{compares } \updownarrow \\
&\ \ r + r \\
&= \ \leftarrow \text{compares } \updownarrow \\
&\ \ 2r \\
&= 2 \cdot \tfrac{1}{2} d(x^*, y^*) \\
&= d(x^*, y^*).
\end{aligned}
$$

That is: $d(x^*, y^*) < d(x^*, y^*)$. $\lightning$

$$2 \le 1 + 3 < 1 + 4 = 5 = 2 + 3$$

Def We say that y_n is a subsequence of x_n if there exists an strictly increasing sequence k_n such that $y_n = x_{k_n}$.

In other words y_n can be obtained by starting from x_n, and deleting some items, eg every second item.

Theorem If $x_n \to x^*$ and y_n is a subsequence of x_n, then $y_n \to x^*$.

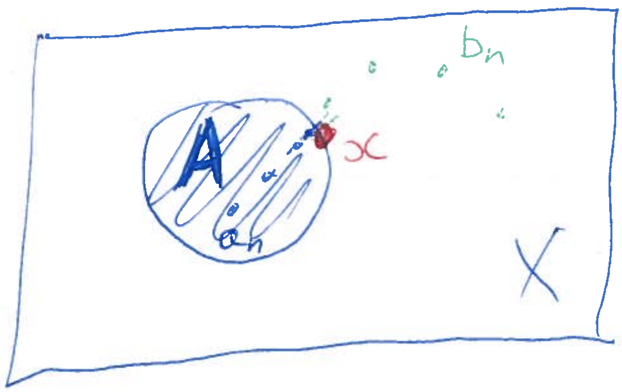
C3 Boundaries

<u>Def</u> Let A be any subset of a metric space (X, d). That is, $A \subseteq X$

A point $x \in X$ is a <u>boundary point</u> of A if

(i) there exists a sequence $a_n \in A$ such that $a_n \to x$, and (nearly inside)

(ii) there exists a sequence $b_n \in \underbrace{X \setminus A}_{\text{"outside of } A\text{"}}$ such that $b_n \to x$. (nearly outside)



<u>Def</u> The set of boundary points of A is called the <u>boundary</u> of A, and is written ∂A.

Examples

* In $(\mathbb{R}, d_2)$, $\partial [0,1] = \{0, 1\}$.

  Let's check $0 \in \partial[0,1]$: $a_n = 0$, $b_n = -\frac{1}{n}$.

* In $(\mathbb{R}, d_2)$, $\partial(0,1) = \{0, 1\}$.
  Let's check $0 \in \partial(0,1)$: $a_n = \frac{1}{n}$, $b_n = 0$.
* In $([0,1], d_2)$, $\partial[0,1] = \emptyset$.
  Let's check $0 \notin \partial[0,1]$: $a_n = 0$, $b_n = ?$
  $X \setminus A = \emptyset$ in the def of ∂
* In $([0,1], d_2)$, $\partial(0,1) = \{0,1\}$.
  Let's check $0 \in \partial(0,1)$: $a_n = \frac{1}{n}$, $b_n = 0$.
* In $(\mathbb{R}_+, d_2)$, $\partial[0,1] = \{1\}$.
  Let's check $1 \in \partial[0,1]$: $a_n = 1$, $b_n = 1 + \frac{1}{n}$

~~C4 Closed Sets~~

* In $(\mathbb{R}, d_2)$, $\partial(\underbrace{[0,1) \cup (1,2]}_{A}) = \{0, 1, 2\}$.
  Let's check: $1 \in \partial A$. $a_n = 1 + \frac{1}{n}$, $b_n = 1$.

C4 Closed Sets

<u>Def</u> Suppose A is a subset of (X, d).
We say A is a <u>closed set</u> if
there is no sequence $a_n \in A$ such
that $a_n \to a^*$ and $a^* \notin A$.

<u>Examples</u>:
* In $(\mathbb{R}, d_2)$: $[0,1]$ is closed.
* In (X, d), $\emptyset$ and X are closed.

* In $((0,1), d_2)$, $(0,1)$ is closed.

But in $(\mathbb{R}, d_2)$, $(0,1)$ is <u>not</u> closed.

eg: $a_n = \frac{1}{n+2}$

Theorem Suppose A is a subset of (X, d). Then A is a closed set if and only if $\partial A \subseteq A$.

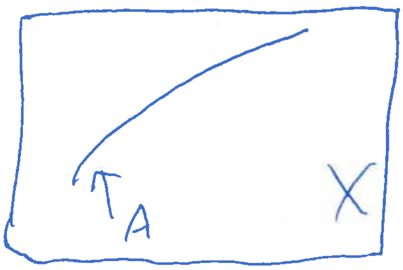
Proof: closed $\Rightarrow \partial A \subseteq A$:

Pick any $x \in \partial A$. ~~Since~~ By the def of boundary, there is some sequence $a_n \in A$ with $a_n \to x$. Since A is closed, and x is the limit of $a_n \in A$, we conclude $x \in A$.

$\partial A \subseteq A \Rightarrow$ closed:

~~Pick~~ Suppose $a_n \to x$ and $a_n \in A$.
Want to prove: $x \in A$.
If this were false, then $b_n = x \in X \setminus A$.
That would imply $x \in \partial A$. But $\partial A \subseteq A$ and so $x \in A$. $\lightning$

eg:



2.1 Production functions

N goods

1 output good

$N-1$ inputs

$f : \mathbb{R}_+^{N-1} \to \mathbb{R}_+$ production function

$y = f(x)$ output quantity

$x \in \mathbb{R}_+^{N-1}$ input quantities $x \in \mathbb{R}_+^{N-1}$

~~Assump~~

Possible assumptions:

* Possibility of inaction: $f(0) = 0$.
* Free disposal(~~m~~ (monotonicity):
  If $x \geq x'$ (that is, $x_n \geq x_n'$ for all $n \in \{1, \dots, N-1\}$) then $f(x) \geq f(x')$.

  monotonicity: either always increasing or always decreasing.

  Can say "monotone increasing".
* Smooth: f is twice differentiable.