G Cont'd

Recall:

$$F(\hat{V})(k) = \sup_{x, k' \geq 0} u(x) + \beta \hat{V}(k')$$
$$\text{s.t.} \quad x + k' = k.$$

Lemma G.1 (Blackwell's Lemma)

Suppose u is bounded. Then F is a contraction of degree β on $(B(\mathbb{R}_+), d_\infty)$.

Proof: Fix any $V \in B(\mathbb{R}_+)$. First we prove $F(V) \in B(\mathbb{R}_+)$. Since u and V are bounded, $\sup_k F(V)(k) \leq \sup_x u(x) + \beta \sup_{k'} V(k')$.

So F is bounded.

~~So~~ Next, we show F is a contraction. Pick any $V_1, V_2 \in B(\mathbb{R}_+)$. Then,

$$F(V_1)(k) = \sup_{x \in [0,k]} u(x) + \beta V_1(k-x)$$
$$= \sup_{x \in [0,k]} u(x) + \underbrace{\beta V_2(k-x) - \beta V_2(k-x)}_{=0} + \beta V_1(k-x)$$
$$\leq \left[\sup_{x \in [0,k]} u(x) + \beta V_2(k-x)\right] + \left[\sup_{x \in [0,k]} -\beta V_2(k-x) + \beta V_1(k-x)\right]$$
$$= F(V_2)(k) + \beta \sup_{x \in [0,k]} \left[V_1(k-x) - V_2(k-x)\right]$$

$$= F(V_2)(k) + \beta \sup_{k' \in [0,k]} [V_1(k') - V_2(k')]$$

$$\leq F(V_2)(k) + \beta \sup_{k' \in [0,k]} |V_1(k') - V_2(k')|$$

$$= F(V_2)(k) + \beta d_\infty(V_1, V_2).$$

We proved: $F(V_1)(k) \leq F(V_2)(k) + \beta d_\infty(V_1, V_2)$

Rearranging, we get

$$F(V_1)(k) - F(V_2)(k) \leq \beta d_\infty(V_1, V_2).$$

Swapping the role of V_1 and V_2, we get

$$F(V_2)(k) - F(V_1)(k) \leq \beta d_\infty(V_1, V_2).$$

So $|F(V_1)(k) - F(V_2)(k)| \leq \beta d_\infty(V_1, V_2)$.

Since this is true for all k, we conclude $d_\infty(F(V_1), F(V_2)) \leq \beta d_\infty(V_1, V_2)$.

So F is a contraction of degree β. $\square$

C9 Compact Sets

Def Let A be a subset of a metric space (X,d). We say A is *compact* if every sequence $x_n \in A$ has a convergent subsequence $y_n \to y^*$ whose limit $y^* \in A$.
We say (X,d) is a *compact metric space* if X is a compact set inside (X,d).

Def A set is *bounded* if it is contained in some open ball.

Theorem (Bolzano–Weierstrass)
Let A be a subset of $(\mathbb{R}^n, d_2)$.
Then A is compact if and only if A is closed and bounded.

is not compact.

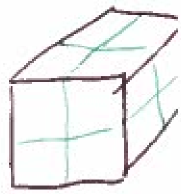
$\mathbb{R}$ is not compact because $x_n = n$ has no convergent subsequence.

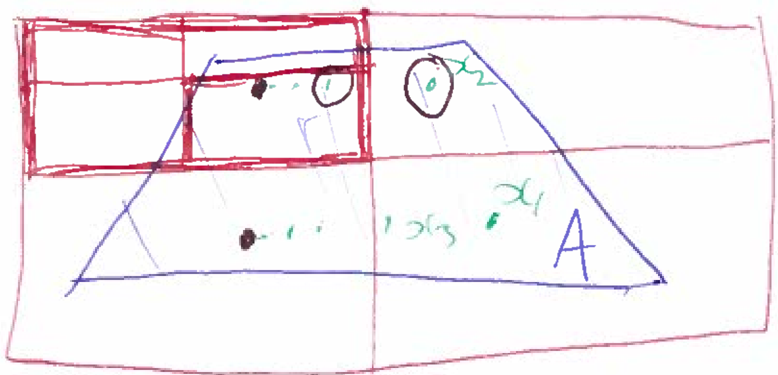
Proof First, we prove A is compact $\Rightarrow$ A is closed & bounded.

Suppose A were not bounded. Then there must be a sequence $x_n \in A$ such that $d(x_0, x_n) > n$. So x_n must be an unbounded sequence, and all of its subsequences are unbounded. So each subsequence is not convergent. But this contradicts A being compact.

Pick any convergent sequence $x_n \in A$ with $x_n \to x^*$. We want to prove $x^* \in A$. Since A is compact, x_n has a convergent subsequence $y_n \to y^* \in A$. Since y_n is a subsequence of x_n, $x^* = y^*$. We conclude $x^* \in A$.

Closed & bounded $\Rightarrow$ compact:





We construct a Cauchy subsequence y_n of x_n (see diagram). Recall $(\mathbb{R}^n, d_2)$ is a complete metric space. So y_n converges to some point $y^* \in \mathbb{R}^n$. Since $y_n \in A$ and A is closed, we conclude $y^* \in A$.

Examples of compact sets:

* $[0,1]$ in $(\mathbb{R}, d_2)$
* $[0,1]^2$ in $(\mathbb{R}^2, d_2)$
* $[0,1] \cup [2,3]$ in $(\mathbb{R}, d_2)$.

<u>Not</u> compact:

* $(0,1)$ in $(\mathbb{R}, d_2)$ — not closed
* $(0,1)$ in $((0,1), d_2)$, even though $(0,1)$ is closed & bounded!
* $\mathbb{R}$ in $(\mathbb{R}, d_2)$ — not bounded.

$x_n = \frac{1}{n}$ has no convergent subsequence

4.3 Equilibrium

Def (Pure-exchange equilibrium)

Consider an economy consisting of a utility function $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ for each household $h \in H$, and an endowment $e_h \in \mathbb{R}^N_+$ for each household. We say that $(x^*, p^*) \in \mathbb{R}^{N \cdot H}_+ \times \mathbb{R}^N$ is ~~an~~ a <u>pure-exchange equilibrium</u> if:

(i) $$x^*_h \in \underset{x_h \in \mathbb{R}^N_+}{\arg\max}\; u_h(x_h) \quad \text{s.t. } p^* \cdot x_h \leq p^* \cdot e_h$$

for all $h \in H$,

(ii) $$\underbrace{\sum_h x^*_{hn}}_{\text{aggregate demand of good } n} = \underbrace{\sum_h e_{hn}}_{\text{aggregate supply of good } n} \quad \text{for all } n.$$

market clearing