Theorem 2.3 For every production function f, the firm's profit function π is convex. Hence if π is smooth,

$$\frac{\partial y(p;w)}{\partial p} \geq 0 \quad \text{and} \quad \frac{\partial x_i(p;w)}{\partial w} \leq 0.$$

Proof Recall

$$\pi(p;w) = \max_{x \in \mathbb{R}^n_+} pf(x) - w \cdot x$$

$$= \max_{x \in \mathbb{R}^n_+} \underbrace{(p,w)\cdot(f(x),-x)}_{\text{Linear in prices}}$$

Recall linear functions are convex. So π is the upper envelope of a set of linear functions (one function per x). So π is ~~concave~~ convex.

Recall that, by the envelope theorem,

$$\frac{\partial \pi(p;w)}{\partial p} = y(p;w)$$

and

$$\frac{\partial \pi(p;w)}{\partial w_i} = -x_i(p;w).$$

Differentiating both sides gives:

$$\frac{\partial^2 \pi(p;w)}{\partial p^2} = \frac{\partial y(p;w)}{\partial p} \quad \text{and} \quad \frac{\partial^2 \pi(p;w)}{\partial w_i^2} = \frac{-\partial x_i(p;w)}{\partial w_i}.$$

Since π is convex, the left sides of both equations are positive. So we conclude

$$\frac{\partial y(p;w)}{\partial p} \geq 0 \quad \text{and} \quad \frac{\partial x_i(p;w)}{\partial w_i} \leq 0. \quad \square$$

2.4 Dynamic Programming

Recall $\pi(p;w) = \max_{x \in \mathbb{R}^{n-1}_+} pf(x) - w \cdot x.$

Cost function:

$$c(y;w) = \min_{x \in \mathbb{R}^{n-1}_+} w \cdot x \quad \text{s.t. } f(x) \geq y.$$

output target

$$\pi(p;w) = \max_{y \in \mathbb{R}_+} py - c(y;w).$$

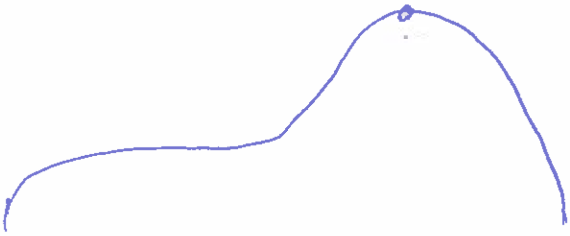
Bellman equation

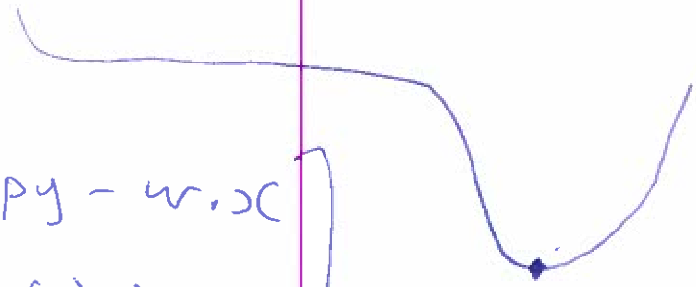
Principle of Optimality: Is the Bellman equation true?

$$\max_{y \in \mathbb{R}_+} py - c(y;w)$$

$$= \max_{y \in \mathbb{R}_+} py - \left[\min_{x \in \mathbb{R}^{n-1}_+} w \cdot x \quad \text{s.t. } f(x) \geq y \right]$$

$$= \max_{y \in \mathbb{R}_+} py + \left[\max_{x \in \mathbb{R}^{n-1}_+} -w \cdot x \quad \text{s.t. } f(x) \geq y \right]$$





$$= \max_{y \in \mathbb{R}_+} \left[\begin{array}{ll} \max\limits_{x \in \mathbb{R}^{n-1}_+} & py - w \cdot x \\ \text{s.t.} & f(x) \geq y \end{array} \right]$$

$$= \max_{x \in \mathbb{R}^{n-1}_+,\, y \in \mathbb{R}_+} py - w \cdot x \quad \text{s.t.} \quad f(x) \geq y$$

$$= \max_{x \in \mathbb{R}^{n-1}_+,\, y \in \mathbb{R}_+} py - w \cdot x \quad \text{s.t.} \quad f(x) = y$$

$$= \max_{x \in \mathbb{R}^{n-1}_+,\, y \in \mathbb{R}_+} pf(x) - w \cdot x \quad \text{s.t.} \quad f(x) = y$$

$$= \max_{x \in \mathbb{R}^{n-1}_+} pf(x) - w \cdot x$$

$$= \pi(p; w). \qquad \square$$

<u>Theorem 2.4</u> For all prices (p, w),

$$p = \left. \frac{\partial c(y; w)}{\partial y} \right|_{y = y(p; w)}.$$

<u>Proof</u> The supply function solves

$$\max_{y \in \mathbb{R}_+} py - c(y; w).$$

So the F.O.C. is: $\frac{\partial}{\partial y} \{ py - c(y; w) \} = 0.$ $\square$

3.2 Cake eating problem

$$V(k) = \max_{x_1,\dots,x_T} u_1(x_1)+u_2(x_2)+\cdots+u_T(x_T)$$

$$\text{s.t. } x_1+\cdots+x_T = k.$$

k ~~[illegible]~~ size of cake

x_t cake consumption on day t

u_t utility function on day t (e.g. birthday)

$$V_t(k_t) = \max_{x_t,\dots,x_T} u_t(x_t)+\cdots+u_T(x_T)$$

$$\text{s.t. } x_t+\cdots+x_T = k_t.$$

size of cake at start of day t

Bellman equation: $V_t(k_t) = \max_{x_t, k_{t+1}\geq 0} u_t(x_t)+V_{t+1}(k_{t+1})$

$$\text{s.t. } x_t + k_{t+1} = k_t.$$

HW: Read the proof of the Principle of Optimality.

G Infinite horizon cake-eating

$$V_t(k) = \sup_{\{x_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(x_s)$$

$$\text{s.t. } \sum_{s=t}^{\infty} x_s = k.$$

Bellman equation: k' means "tomorrow"

$$V(k) = \sup_{x,k'\geq 0} u(x) + \beta V(k')$$

$$\text{s.t. } x+k' = k.$$

Bellman operator ∞ eq's | ∞ unknowns

$$F(\hat{V})(k) = \sup_{x, k' \geq 0} u(x) + \beta \hat{V}(k)$$
$$\text{s.t. } x + k' = k$$

$F: B(\mathbb{R}_+) \to B(\mathbb{R}_+)$.

e.g. Let $V_0(k) = 0$.

Then $F(V_0)(k) = u(k)$.

Is there a solution V^* to the Bellman equation: $V^* = F(V^*)$?

C.8 Fixed points

Def A function f is a <u>self-map</u> if $f: X \to X$, i.e. domain(f) = co-domain(f).

Def Let $f: X \to X$ be a self-map. A point $x^* \in X$ is called a <u>fixed point of f</u> if $x^* = f(x^*)$.

Is there a value function V^* that is a fixed point of the Bellman operator F?

<u>Def</u> Let (X, d_X) and (Y, d_Y) be metric spaces, and $a>0$. We say $f: X \to Y$ is <u>Lipschitz continuous of degree a</u> if for every $x, x' \in X$,

$$d_Y(f(x), f(x')) \le a\, d_X(x, x').$$

<u>Theorem</u> If f is Lipschitz continuous, then f is continuous.

<u>Def</u> Let (X, d) be a metric space. The self-map $f: X \to X$ is called a <u>contraction</u> if it is Lipschitz continuous of degree $a<1$.

<u>Theorem (Banach's fixed point theorem)</u> Let (X, d) be a complete metric space. If $f: X \to X$ is a contraction of degree a, then

① f has a unique fixed point x^*.

② Given any $x_0 \in X$, the sequence $x_{n+1} = f(x_n)$ converges to x^*.

③ $d(x_n, x^*) \le \frac{a^n}{1-a} d(x_0, x_1)$.

<u>Proof</u> <u>Uniqueness</u> Suppose x^* and x^{**} are both fixed points of f.

Since they are fixed points,

$$d(\underbrace{f(x^*)}_{= x^*}, \underbrace{f(x^{**})}_{= x^{**}}) = d(x^*, x^{**}).$$

But since f is a contraction,

$$d(f(x^*), f(x^{**})) \leq a\, d(x^*, x^{**}).$$

So $x^* = x^{**}$.

<u>Existence and convergence</u>

We plan prove x_n is a Cauchy sequence.

Repeated application of the contraction property gives apply f n times

$$d(x_n, x_{n+m}) = d(f^n(x_0), f^n(x_m)) \leq a^n\, d(x_0, x_m).$$

This implies:

triangle inequality

$$\begin{aligned} d(x_0, x_m) &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m) \\ &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots \quad \text{∞ number of terms} \\ &\leq d(x_0, x_1) + a\, d(x_0, x_1) + a^2 d(x_0, x_1) + \cdots \\ &= [\underbrace{1 + a + a^2 + \cdots}_{\text{geometric series}}]\, d(x_0, x_1) \\ &= \frac{1}{1-a}\, d(x_0, x_1). \end{aligned}$$

Combining these two properties, we get

$$d(x_n, x_{n+m}) \le \frac{a^n}{1-a} d(x_0, x_1) \text{ for all } n, m,$$

which we reformulate as

$$\circledast \quad d(x_n, x_m) \le \frac{a^N}{1-a} d(x_0, x_1) \text{ for all } n, m \ge N$$

So x_n is a Cauchy sequence.

Since x_n is a Cauchy sequence and (X, d) is complete, x_n converges to some point x^*. Since f is continuous, $y_n = f(x_n)$ converges to $f(x^*)$. But $y_n = x_{n+1}$, so y_n is a subsequence of x_n, so $y_n \to x^*$. So $y_n \to f(x^*)$ and $y_n \to x^*$, so $x^* = f(x^*)$. We conclude x^* is a fixed point of f.

Guess quality formula: by $\circledast$

$$d(x_n, x^*) = \lim_{m\to\infty} d(x_n, x_m) \le \lim_{m\to\infty} \frac{a^n}{1-a} d(x_0, x_1)$$

since d is continuous

$$= \frac{a^n}{1-a} d(x_0, x_1).$$

□